



Review

# Geometric transitions<sup>☆</sup>

Michele Rossi

*Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy*

Received 1 February 2005; accepted 9 September 2005

Available online 14 November 2005

---

## Abstract

The purpose of this paper is to give, on one hand, a mathematical exposition of the main topological and geometrical properties of geometric transitions, on the other hand, a quick outline of their principal applications, both in mathematics and in physics.

© 2005 Elsevier B.V. All rights reserved.

*Keywords:* Geometric transitions; Conifold transitions; Mirror symmetry; Reid’s fantasy; Vacuum degeneracy problem; Reverse transition; Toric degenerations; Moduli spaces of Calabi–Yau three-folds

---

## Contents

1. Geometric transitions: definition and the basic example .....	1942
1.1. Calabi–Yau varieties .....	1942
1.2. Geometric transitions .....	1943
1.3. The basic example: the conifold in $\mathbb{P}^4$ .....	1944
2. Local geometry and topology of a conifold transition .....	1946
2.1. The local topology of a node .....	1946
2.2. Local geometry of the resolution .....	1947
2.3. Local geometry of the smoothing .....	1949
2.4. Local topology of a conifold transition .....	1950
3. Global geometry and topology of a conifold transition .....	1950
3.1. What about more general geometric transitions? .....	1959

---

<sup>☆</sup> Research partially supported by the Italian PRIN project “Geometria delle Varietà Algebriche” (GVA) and by the Dipartimento di Matematica dell’Università di Torino (local research grant: “Varietà Toriche, transizioni geometriche e dualità fisiche”).

*E-mail address:* [michele.rossi@unito.it](mailto:michele.rossi@unito.it).

4. Classification of geometric transitions . . . . .	1960
4.1. A little bit of Mori theory for Calabi–Yau three-folds . . . . .	1960
4.2. Primitive contractions and primitive transitions . . . . .	1962
4.3. Smoothing the target space $\tilde{Y}$ . . . . .	1963
4.3.1. Transitions of type I . . . . .	1963
4.3.2. Transitions of type II . . . . .	1965
4.3.3. Transitions of type III . . . . .	1966
4.3.4. What about a general transition? . . . . .	1967
5. The Calabi–Yau web . . . . .	1967
5.1. Reid’s fantasy . . . . .	1967
5.2. The “vacuum degeneracy problem” in string theory . . . . .	1970
5.3. The connectedness conjecture . . . . .	1970
6. Mirror symmetry and transitions: the reverse transition . . . . .	1973
6.1. Mirror symmetry conjecture: some mathematical statements . . . . .	1973
6.1.1. The Calabi–Yau moduli space . . . . .	1974
6.2. The reverse transition . . . . .	1976
6.3. Toric degenerations: conifold transitions to construct mirror manifolds . . . . .	1978
6.4. Mirror partners of rigid Calabi–Yau three-folds via geometric transitions . . . . .	1979
7. Further physical dualities and transitions . . . . .	1980
Acknowledgements . . . . .	1980
References . . . . .	1980

A geometric transition is a birational contraction followed by a complex smoothing. This process connects two smooth, topologically distinct, Calabi–Yau three-folds. For this reason geometric transitions attracted the interest of both mathematician and physicists.

From the mathematical point of view, the property of changing topology candidates geometric transitions as the three-dimensional analogous of analytic deformations between  $K3$  surfaces. More precisely,  $K3$  projective surfaces having different sectional genus are linked by analytic deformations, showing that their moduli space is actually connected. Analogously geometric transitions may be the right way to give a notion of “connectedness” to the “moduli space” of Calabi–Yau three-folds. This is essentially the famous Reid’s fantasy [62] founded on deep speculations due to Clemens [21], Friedman [27], Hirzebruch [38] and Werner [77].

On the other hand, in physics, the same property provides a mathematical tool to connect topologically distinct compactifications to four dimensions of 10-dimensional type II super-string theory vacua. This fact was firstly observed by Candelas et al. [17,31,32,18,19]. The physical interpretation of a geometric transition connecting two topologically distinct string vacua was given later, in 1995, by Strominger [68], at least in the case of a *conifold* transition i.e. a geometric transition whose associated birational contraction generates at most ordinary double points. After this pivotal paper other geometric transitions have been physically understood [12,41,13].

For many geometric transitions, the induced change in topology can be summarized by saying that *a transition increases complex moduli and decreases Kähler moduli*. Since mirror symmetry exchange complex and Kähler moduli, it seemed natural to conjecture the existence of a *reverse transition* connecting mirror partners of a couple of Calabi–Yau three-folds linked by a given transition [51]. Reverse transitions have been then revealed useful tools for producing, at least conjecturally, mirror constructions extending, via *toric degenerations*, the Batyrev mirror symmetry between Calabi–Yau three-folds embedded in toric varieties [8,9,7].

In physics, geometric transitions have newly been in the spot light as the geometric set up of recently conjectured open/closed string dualities [29,56].

The present work is meant to give on one hand a mathematical exposition of the main topological and geometrical properties of a transition. This is the program of Sections 1–4: except for the latter, where some notion of deformation theory in geometry is needed, these sections are devoted to present a, as much as possible, self-contained treatment, for graduate students and beginners. For this reason many well-known results or properties are developed in details like some example (see Section 1.3 and Example 3.1) or Theorem 3.3. In particular the latter is intended to give a complete account of the change in topology induced by a conifold transition. Its content was already known 20 years ago to Clemens, and then to many other mathematicians and physicists, but I was not able to find, in the literature, a complete statement and a clear proof of all of the results mentioned there. For this reason I preferred to rewrite here an elementary proof requiring no more than basic facts in algebraic topology and geometry.

On the other hand Sections 5–7 give a quick outline of some applications of geometric transitions both in mathematics and in physics. Here the reader is clearly required to know basic facts and definitions of these topics, although I tried to give references of the original papers and, sometimes, of extended surveys treating the mentioned subjects.

The paper is organized as follows.

Section 1 is devoted to give definition and examples of geometric transitions. In particular the fundamental example of a non-trivial conifold transition involving a quintic three-fold in  $\mathbb{P}^4$  is developed in detail.

Section 2 is a revised version of some of the “topological considerations” given by Clemens [21], which allow to locally think a conifold transition as a surgery in topology (Proposition 2.10).

In Section 3 the global change in topology induced by a conifold transition is carefully studied, relying each other homological invariants of all of the three poles of a conifold transition (Theorem 3.3). This section ends up with some similar considerations for more general geometric transitions, essentially due to Namikawa and Steenbrink [55].

Section 4 gives an outline of results and techniques needed to perform a (actually incomplete) classification of geometric transition. Main results are here due to Friedman, Gross and Namikawa.

The remaining sections are dedicated to describe some fundamental applications of geometric transitions. Section 5 describes how geometric transitions are conjecturally employed, in mathematics, to think of the Calabi–Yau three-folds moduli space as “irreducible” and, in physics, to “unify” type II super-string compactified vacua. In Section 6 a quick account of the role played by geometric transitions in mirror symmetry is given, starting from the key concept of reverse transition. In Section 7 some further more recent applications are finally mentioned.

## 1. Geometric transitions: definition and the basic example

### 1.1. Calabi–Yau varieties

**Definition 1.1.** Let  $Y$  be a smooth, complex, projective variety with  $\dim Y \geq 3$ .  $Y$  will be called a *Calabi–Yau variety* if

- (1)  $\wedge^n \Omega_Y =: \mathcal{K}_Y \cong \mathcal{O}_Y$ ,
- (2)  $h^{p,0}(Y) = 0 \quad \forall 0 < p < \dim Y$ .

A three-dimensional Calabi–Yau variety will be also called a *Calabi–Yau three-fold*.

**Remark 1.2.**

(1) There are a lot of more or less equivalent definitions of Calabi–Yau varieties coming from:

- *differential geometry*: the differential geometric concept of a compact, Kähler manifold admitting a *Ricci flat metric* (Calabi conjecture and Yau theorem),
- *theoretical physics*: the physical concept of a Kähler, three-dimensional complex, compact manifold admitting a flat, non-degenerate, holomorphic three-form.

(see [40] for a complete description of equivalences and implications).

(2) In the algebraic context, the given definition of Calabi–Yau variety is the generalization of the following geometric objects

- *One-dimensional*: smooth elliptic curves,
- *Two-dimensional*: smooth K3 surfaces.

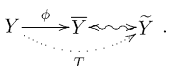
(3) With the dimensional bound  $\dim Y \geq 3$ , the given definition of Calabi–Yau variety is equivalent to require that  $Y$  is a *Kähler, compact, manifold whose holonomy group is a subgroup of  $SU(\dim Y)$*  (cf. [40]).

**Example 1.3.**

- (1) Smooth hypersurfaces of degree  $n + 1$  in  $\mathbb{P}^n$  (use adjunction formula and the Lefschetz hyperplane theorem).
- (2) Smooth hypersurfaces (if exist!) of a weighted projective space  $\mathbb{P}(q_0, \dots, q_n)$  of degree  $d = \sum_{i=0}^n q_i$ .
- (3) The general element of the anti-canonical system of a *sufficiently good* four-dimensional toric Fano variety (see [5]).
- (4) Suitable complete intersections ... (iterate the previous examples).
- (5) The double covering of  $\mathbb{P}^3$  ramified along a smooth surface of degree 8 in  $\mathbb{P}^3$  (octic double solid).

1.2. Geometric transitions

**Definition 1.4** (cf. [51,22,30]). Let  $Y$  be a Calabi–Yau three-fold and  $\phi : Y \rightarrow \bar{Y}$  be a *birational contraction* onto a *normal* variety. If there exists a complex deformation (*smoothing*) of  $\bar{Y}$  to a Calabi–Yau three-fold  $\tilde{Y}$ , then the process of going from  $Y$  to  $\tilde{Y}$  is called a *geometric transition* (for short *transition*) and denoted by  $T(Y, \bar{Y}, \tilde{Y})$  or by the diagram



A transition  $T(Y, \bar{Y}, \tilde{Y})$  is called *trivial* if  $\tilde{Y}$  is a deformation of  $Y$ .

**Remark 1.5.**

- (1) *Trivial transitions may occur:* e.g. consider Example 4.6 in [79], where  $\phi$  admits an elliptic scroll as exceptional divisor and contracts it down to an elliptic curve  $C$ .
- (2) It is clearly possible to extend the transition process to any dimension  $\geq 3$ . Note that it is not possible to realize non-trivial transitions in dimension 1 (i.e. between elliptic curves).
- (3) The transition process was firstly (locally) observed by Clemens in the study of double solids  $V$  admitting at worst nodal singularities [21]: in his Lemma 1.11 he pointed out “the relation of the resolution of the singularities of  $V$  to the standard  $S^3 \times D_3$  to  $S^2 \times D_4$  surgery”.

**Definition 1.6.** A transition  $T(Y, \bar{Y}, \tilde{Y})$  is called *conifold* if  $\bar{Y}$  admits only *ordinary double points* (nodes) as singularities, i.e. singular points whose tangent cones are singular hyperquadrics of rank  $\dim X + 1$  (precisely *non-degenerate cones*).

1.3. *The basic example: the conifold in  $\mathbb{P}^4$*

The following example, given in [34], shows that *non-trivial (conifold) transitions occur when  $\dim X \geq 3$* .

Let  $\bar{Y} \subset \mathbb{P}^4$  be the singular hypersurface given by the following equation

$$x_3g(x_0, \dots, x_4) + x_4h(x_0, \dots, x_4) = 0 \tag{1}$$

where  $g$  and  $h$  are generic homogeneous polynomials of degree 4.  $\bar{Y}$  is then the *generic quintic three-fold containing the plane  $\pi : x_3 = x_4 = 0$* . Then the singular locus of  $\bar{Y}$  is given by

$$\text{Sing}(\bar{Y}) = \{[x] \in \mathbb{P}^4 \mid x_3 = x_4 = g(x) = h(x) = 0\} \tag{2}$$

**Proposition 1.7.** *Sing( $\bar{Y}$ ) is composed by 16 nodes.*

**Proof.** Let  $p \in \text{Sing}(\bar{Y})$ . We have to write down the local equation of  $p$ . Assume  $p = [1, 0, 0, 0, 0]$  and intersect  $\bar{Y}$  with the affine open subset of  $\mathbb{P}^4$

$$U_0 := \{[x] \in \mathbb{P}^4 \mid x_0 \neq 0\}$$

Set  $z_i := x_i/x_0, i = 1, \dots, 4$ . Then  $\bar{Y} \cap U_0$  is described by the following affine equation

$$z_3\tilde{g}(z) + z_4\tilde{h}(z) = 0 \tag{3}$$

where  $x_0^4\tilde{g} = g$  and  $x_0^4\tilde{h} = h$ . Besides  $p$  is the origin of  $U_0$ .

Since  $g, h$  are generic we can assume that the polynomial (holomorphic) maps  $\tilde{g}, \tilde{h} : \mathbb{C}^4 \rightarrow \mathbb{C}$  are submersive at the origin and we can find a holomorphic chart  $(U, z)$  centered in  $p = 0 \in \mathbb{C}^4$  and such that

$$\bar{U} := \bar{Y} \cap U : z_3z_1 + z_4z_2 = 0 \tag{4}$$

Then  $p$  is a node.  $\square$

**Proposition 1.8** (The resolution). *Sing( $\bar{Y}$ ) can be simultaneously resolved and the resolution  $\phi : Y \rightarrow \bar{Y}$  is a small blow up such that  $Y$  is a smooth Calabi–Yau three-fold.*

**Proof.** Blow up  $\mathbb{P}^4$  along the plane  $\pi : x_3 = x_4 = 0$ . We get a birational morphism

$$\hat{\phi} : \hat{\mathbb{P}}^4 \rightarrow \mathbb{P}^4$$

whose exceptional divisor is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^2$ . Let  $Y$  be the *proper transform* of  $\bar{Y}$  (i.e. the closure in  $\hat{\mathbb{P}}^4$  of  $\hat{\phi}^{-1}(\bar{Y} \setminus \pi)$ ). Since  $\hat{\mathbb{P}}^4$  is the hypersurface of bi-homogeneous equation  $y_0x_4 - y_1x_3 = 0$  in  $\mathbb{P}^4(x) \times \mathbb{P}^1(y)$ , then  $Y$  is the following complete intersection

$$y_0x_4 - y_1x_3 = 0, \quad y_0g(x) + y_1h(x) = 0 \tag{5}$$

and we get that

- $Y$  is smooth,
- $\phi := \hat{\phi}|_Y : Y \rightarrow \bar{Y}$  is an isomorphism outside of  $\text{Sing}(\bar{Y})$ ,
- $\forall p \in \text{Sing}(\bar{Y}), \phi^{-1}(p) \cong \mathbb{P}^1$ .

Hence  $\phi : Y \rightarrow \bar{Y}$  is a birational resolution called *small blow up* due to the dimension of its exceptional locus ( $1 < \dim Y - 1 = 2$ ).

To prove that  $Y$  is Calabi–Yau recall that  $\hat{\phi}$  is a blow up, hence

$$K_{\hat{\mathbb{P}}^4} \equiv \hat{\phi}^*(K_{\mathbb{P}^4}) + (4 - 2 - 1)E \equiv -5\hat{\phi}^*(H) + E$$

where  $E$  is the exceptional divisor of  $\hat{\phi}$  and  $H$  is the hyperplane of  $\mathbb{P}^4$ . Then the adjunction formula gives

$$\mathcal{K}_Y \cong \mathcal{K}_{\hat{\mathbb{P}}^4} \otimes \mathcal{O}_{\hat{\mathbb{P}}^4}(Y) \otimes \mathcal{O}_Y \cong \mathcal{O}_Y(E|_Y) \cong \mathcal{O}_Y$$

Moreover the Lefschetz hyperplane theorem and the Künneth formula give

$$H^1(Y, \mathbb{C}) \cong H^1(\hat{\mathbb{P}}^4, \mathbb{C}) \cong H^1(\mathbb{P}^4 \times \mathbb{P}^1, \mathbb{C}) = 0$$

hence  $h^{1,0}(Y) = 0$ . On the other hand the Serre duality theorem allows to conclude that

$$H^2(Y, \mathcal{O}_Y) \cong H^1(Y, \mathcal{K}_Y) \cong H^1(Y, \mathcal{O}_Y)$$

hence  $h^{2,0}(Y) = h^{0,2}(Y) = h^{0,1}(Y) = h^{1,0}(Y) = 0$ .  $\square$

**Proposition 1.9** (The smoothing).  $\bar{Y}$  admits the obvious smoothing given by the generic quintic three-fold  $\tilde{Y} \subset \mathbb{P}^4$ . In particular  $\tilde{Y}$  cannot be a deformation of  $Y$  i.e. the conifold transition  $T(Y, \tilde{Y}, \bar{Y})$  is not trivial.

**Proof.** Apply again the Lefschetz hyperplane theorem and the Künneth formula to get the following relations on the Betti numbers of  $\tilde{Y}$  and  $Y$

$$b_2(\tilde{Y}) = b_2(\mathbb{P}^4) = 1, \quad b_2(Y) = b_2(\mathbb{P}^4 \times \mathbb{P}^1) = 2 \tag{6}$$

Therefore  $\tilde{Y}$  and  $Y$  cannot be smooth fibres of the same analytic family.  $\square$

## 2. Local geometry and topology of a conifold transition

The present section will be essentially devoted to explain the basic argument given by Clemens [21]. As a consequence we get that *locally a conifold transition is described by a suitable surgery*.

In this section we will always assume that  $T(Y, \bar{Y}, \tilde{Y})$  is a *conifold transition* and  $p$  is a point in  $\text{Sing}(\bar{Y})$ , which means that it is a *node*.

### 2.1. The local topology of a node

Just like in the basic [Example 1.3](#), we may assume that there exists a local chart  $(U, z)$  such that  $p = 0 \in U$ . Denote  $\bar{U} := \bar{Y} \cap U$ , which has local equation in  $U$  given by

$$z_1 z_3 + z_2 z_4 = 0 \tag{7}$$

**Proposition 2.1.** *Topologically  $\bar{U}$  is a cone over  $S^3 \times S^2$ .*

**Proof.** Change coordinates as follows

$$\begin{aligned} w_1 &= \frac{1}{2}(z_1 + z_3), & w_2 &= \frac{i}{2}(-z_1 + z_3), \\ w_3 &= \frac{1}{2}(z_2 + z_4), & w_4 &= \frac{i}{2}(-z_2 + z_4) \end{aligned} \tag{8}$$

to rewrite the local equation (7) as

$$\sum_{j=1}^4 w_j^2 = 0$$

Decompose the latter in real and imaginary parts by setting  $w_j = u_j + iv_j$ . Then  $\bar{U}$  is described in  $\mathbb{R}^8(u, v)$  by the following two equations

$$\sum_{j=1}^4 u_j^2 - \sum_{j=1}^4 v_j^2 = 0 \tag{9}$$

$$\sum_{j=1}^4 u_j v_j = 0 \tag{10}$$

Fix now a real positive radius  $\rho$  and consider the seven-sphere

$$S_\rho^7 := \left\{ (u, v) \in \mathbb{R}^8 \mid \sum_{j=1}^4 u_j^2 + \sum_{j=1}^4 v_j^2 = \rho^2 \right\}$$

Cut then  $\bar{U}$  to get  $\bar{U}_\rho := \bar{U} \cap S_\rho^7$ . Topologically  $\bar{U} = \bigsqcup_{\rho \geq 0} \bar{U}_\rho$  and we get the claim by proving that  $\bar{U}_\rho \cong S^3 \times S^2$ .

At this purpose, note that  $\bar{U}_\rho$  is described in  $\mathbb{R}^8$  by the following equations

$$\sum_{j=1}^4 u_j^2 = \rho^2 - \sum_{j=1}^4 v_j^2, \quad \sum_{j=1}^4 v_j^2 = \frac{\rho^2}{2}, \quad \sum_{j=1}^4 u_j v_j = 0 \tag{11}$$

Then  $\tilde{U}_\rho$  can be fibred over the three-sphere  $S^3_{\rho/\sqrt{2}} := \{v \in \mathbb{R}^4 \mid \sum_{j=1}^4 v_j^2 = \rho^2/2\}$ . Precisely the fibre over a point  $v^o \in S^3_{\rho/\sqrt{2}}$  is given by

$$\sum_{j=1}^4 u_j^2 = \frac{\rho^2}{2}, \quad \sum_{j=1}^4 v_j^o u_j = 0$$

which is a two-sphere of radius  $\rho/\sqrt{2}$ .

The proof ends up by showing that the bundle  $\tilde{U}_\rho$  is actually a product. This fact follows by observing that  $\tilde{U}_\rho$  is embedded in the tangent bundle to the three-sphere  $S^3_{\rho/\sqrt{2}} \subset \mathbb{R}^4(v)$ . In fact the latter is embedded in  $\mathbb{R}^8(u, v)$  by the second and third equations in (11). To conclude restrict to  $\tilde{U}_\rho$  the well-known trivialization  $TS^3_{\rho/\sqrt{2}} \cong S^3 \times \mathbb{R}^3$ .  $\square$

### 2.2. Local geometry of the resolution

To resolve the node recall Proposition 1.8 of the basic example. Precisely look at the proper transform  $\hat{U}$  of  $\tilde{U}$  in the blow up of the local chart  $(U, z) \cong \mathbb{C}^4(z)$  along the plane  $z_3 = z_4 = 0$ .  $\hat{U}$  is then described in  $\mathbb{C}^4 \times \mathbb{P}^1$  by the following equations

$$y_0 z_4 - y_1 z_3 = 0, \quad y_0 z_1 + y_1 z_2 = 0 \tag{12}$$

**Proposition 2.2.** *There is a diffeomorphism  $\hat{U} \cong \mathbb{R}^4 \times S^2$*

**Proof.** Topologically it is not difficult to observe that  $\hat{U}$  is an  $\mathbb{R}^4$ -bundle over  $\mathbb{P}^1_{\mathbb{C}}$ . In fact by splitting  $z_j$  in real and imaginary parts, Eq. (12) give rise to four linear equations in  $\mathbb{R}^8$  parameterized by  $[y_0, y_1] \in \mathbb{P}^1_{\mathbb{C}}$ .

To construct the diffeomorphism introduce the coordinates change given by (8) and split the new coordinates in real and imaginary parts:  $w_j = u_j + iv_j$ . Eq. (12) of  $\hat{U}$  can then be rewritten in  $\mathbb{R}^8(u, v) \times \mathbb{P}^1_{\mathbb{C}}$  in the following matricial form

$$\mathbf{u} = A([y_0, y_1])\mathbf{v} \tag{13}$$

where

$$A([y_0, y_1]) := \begin{pmatrix} 0 & |y_0|^2 - |y_1|^2 & 2 \operatorname{Im}(\bar{y}_0 y_1) & 2 \operatorname{Re}(\bar{y}_0 y_1) \\ -|y_0|^2 + |y_1|^2 & 0 & -2 \operatorname{Re}(\bar{y}_0 y_1) & -2 \operatorname{Im}(\bar{y}_0 y_1) \\ -2 \operatorname{Im}(\bar{y}_0 y_1) & 2 \operatorname{Re}(\bar{y}_0 y_1) & 0 & -|y_0|^2 + |y_1|^2 \\ -2 \operatorname{Re}(\bar{y}_0 y_1) & 2 \operatorname{Im}(\bar{y}_0 y_1) & |y_0|^2 - |y_1|^2 & 0 \end{pmatrix} \tag{14}$$

We will refer to the matrix  $A$  as the *Clemens' matrix*: in fact it is the same matrix appearing in formula (1.18) of [21]. For any  $[y] \in \mathbb{P}^1_{\mathbb{C}}$ , one can easily check that  $A[y] \in SO(4)$  and moreover it is antisymmetric i.e.  ${}^t A[y] + A[y] = 0$ .

A diffeomorphism  $\Phi : \hat{U} \cong \mathbb{R}^4 \times \mathbb{P}^1_{\mathbb{C}}$  is then given by

$$\Phi^{-1} : \mathbb{R}^4 \times \mathbb{P}^1_{\mathbb{C}} \rightarrow \hat{U} \subset \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{P}^1_{\mathbb{C}}, \quad (v, [y]) \mapsto (A[y]v, v, [y]) \tag{15}$$

The proof ends up by the usual identification  $\mathbb{P}^1_{\mathbb{C}} \cong S^2$ .  $\square$



**Remark 2.3.** Just like in the basic [Example 1.3](#), the restriction of the blow up of  $U = \mathbb{C}^4$  along the plane  $z_3 = z_4 = 0$  gives rise to a birational map

$$\varphi : \hat{U} \rightarrow \bar{U}$$

which is a *small blow up*. Precisely  $\varphi$  is biregular over the complement of the origin in  $\bar{U}$  and  $\varphi^{-1}(0) = \mathbb{P}_{\mathbb{C}}^1$ . Then it induces a diffeomorphism

$$\hat{U} \setminus \varphi^{-1}(0) \xrightarrow[\cong]{\varphi} \bar{U} \setminus \{0\}$$

Recalling [Proposition 2.1](#),  $\bar{U} \setminus \{0\} \cong (\mathbb{R}^4) \setminus \{0\} \times \mathbb{P}_{\mathbb{C}}^1$  and it is natural to ask what is the relation between  $\varphi$  and  $\Phi$ . Thanks to the Clemens matrix’s properties we get that

$$\Phi|_{\hat{U} \setminus \varphi^{-1}(0)} = \varphi|_{\hat{U} \setminus \varphi^{-1}(0)} \tag{16}$$

and  $\Phi$  is an extension of  $\varphi$  over the exceptional fibre i.e. the following commutative diagram holds

$$\begin{array}{ccc} \hat{U} & \xrightarrow{\Phi} & \mathbb{R}^4 \times S^2 \\ \uparrow & & \uparrow \\ \hat{U} \setminus \varphi^{-1}(0) & \xrightarrow{\varphi} & \bar{U} \setminus \{0\} \end{array}$$

To prove this fact it suffices to check that  $(u, v) = (Av, v)$  satisfies the real equations (9) of  $\bar{U}$ , for any  $v \neq 0$ . In fact

$$|u|^2 - |v|^2 = {}^t v^t A A v - {}^t v v = 0$$

since  $A$  is orthogonal. On the other hand

$$\sum_{j=1}^4 u_j v_j = {}^t v^t A v = -{}^t v A v = 0$$

since  $A$  is antisymmetric and it induces an alternating bilinear form.

**Proposition 2.4.**  $\hat{U}$  can be identified with the total space of the rank 2 holomorphic vector bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  over the exceptional fibre  $\mathbb{P}_{\mathbb{C}}^1 = \varphi^{-1}(0)$ . In particular  $\hat{U}$  admits a natural complex structure.

**Proof.** Since  $\hat{U}$  is the proper transform of  $\bar{U}$  in the small blow up of  $U$ , it can be identified with the total space of the normal bundle of the exceptional fibre  $\mathcal{N}_{\hat{U}|\mathbb{P}^1}$ . The latter is a holomorphic vector bundle of rank 2 over the exceptional fibre  $\mathbb{P}_{\mathbb{C}}^1$ . By the Grothendieck theorem it splits as follows:

$$\mathcal{N}_{\hat{U}|\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$$

Choose two local charts on  $S^2 \cong \mathbb{P}_{\mathbb{C}}^1$  around the north and the south poles, respectively. Let  $\tau := y_0/y_1$  and  $\sigma := y_1/y_0$  be the associated local coordinates. Lifting these charts to  $\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$  means that we can choose two local parameterizations

$$(\tau; t_1, t_2), \quad (\sigma; s_1, s_2)$$

patching along the fibre over the fixed point  $(y_0 : y_1) = (\tau : 1) = (1 : \sigma)$  as follows:

$$s_i = \tau^{-d_i} t_i$$

where  $\tau^{-d_i}$  represents the transition function in  $GL(1, \mathbb{C}) = \mathbb{C}^*$ .

Eq. (12) of  $\tilde{U}$  allow us to set

$$t_1 = z_1, \quad t_2 = z_4; \quad s_1 = -z_2, \quad s_2 = z_3$$

Then

$$s_1 = -z_2 = \frac{y_0}{y_1} z_1 = \tau t_1, \quad s_2 = z_3 = \frac{y_0}{y_1} z_4 = \tau t_2$$

and we get that  $d_1 = d_2 = -1$ .  $\square$

### 2.3. Local geometry of the smoothing

Recalling the real equations (9) of  $\bar{U}$ , a local smoothing of the node is given by the 1-parameter family  $f : \mathcal{U} \rightarrow \mathbb{R}$  where

$$U_t := f^{-1}(t) : \begin{cases} \sum_{j=1}^4 u_j^2 - \sum_{j=1}^4 v_j^2 = t \\ \sum_{j=1}^4 u_j v_j = 0 \end{cases} \tag{17}$$

Let  $\tilde{U} := U_{t_0}$  for some  $t_0 \in \mathbb{R}, t_0 > 0$ .

**Proposition 2.5.**  $\tilde{U}$  is diffeomorphic to the cotangent bundle  $T^*S^3$  of the three-sphere. In particular  $\tilde{U} \cong S^3 \times \mathbb{R}^3$ .

**Proof.**  $T^*S^3$  can be embedded in  $\mathbb{R}^8(q, p)$  by the standard equations

$$\sum_{j=1}^4 q_j^2 = 1, \quad \sum_{j=1}^4 q_j p_j = 0$$

The diffeomorphism  $\Psi : \tilde{U} \cong T^*S^3$  is then defined by setting

$$q_j = \frac{u_j}{\sqrt{t_0 + \sum_{j=1}^4 v_j^2}}, \quad p_j = v_j$$

The proof concludes by applying the standard trivialization  $T^*S^3 \cong S^3 \times \mathbb{R}^3$ .  $\square$

**Remark 2.6.** The vanishing cycle of the smoothing  $f : \mathcal{U} \rightarrow \mathbb{R}$  is given by the family of embedded three-spheres  $\mathcal{S} \rightarrow \mathbb{R}$  defined by

$$S_t := \begin{cases} |u|^2 - t = v_1 = \dots = v_4 = 0 & \text{if } t \geq 0 \\ |v|^2 - t = u_1 = \dots = u_4 = 0 & \text{if } t \leq 0 \end{cases} \tag{18}$$

Clearly  $S_0 = \{0\} \subset \bar{U}$ . Define  $\tilde{\mathcal{S}} := S_{t_0}$ . Recalling the diffeomorphism  $\Psi$  of the previous proposition we get that  $\Psi(\tilde{\mathcal{S}})$  is the 0-section of the cotangent bundle  $T^*S^3$ .

**Definition 2.7.** Let  $L$  be a submanifold of a given symplectic manifold  $(M, \omega)$ .  $L$  is called *La-grangian* if

- (1)  $2 \dim_{\mathbb{R}} L = \dim_{\mathbb{R}} M$
- (2)  $\forall p \in L, \forall X, Y \in T_p M, \omega_p(X, Y) = 0$ .

**Example 2.8.** The cotangent bundle  $T^*M$  of a given manifold  $M$  admits the *canonical* symplectic structure given by  $\omega := d\vartheta$ , where  $\vartheta$  is the Liouville one-form. The 0-section of  $T^*M$  is a Lagrangian submanifold with respect to the canonical symplectic structure.

**Proposition 2.9.**  $\tilde{U}$  admits a natural symplectic structure and the vanishing cycle  $\tilde{S}$  is a Lagrangian submanifold.

**Proof.** Let  $\omega$  be the canonical symplectic structure on  $T^*S^3$ . Then  $\Psi^*(\omega)$  gives the natural symplectic structure to  $\tilde{U}$ . By Remark 2.6 and Example 2.8 we get that

$$\Psi^*(\omega)|_{\tilde{S}} = \omega|_{S^3} = 0 \quad \square$$

2.4. Local topology of a conifold transition

**Proposition 2.10** ([21, Lemma 1.11]). Let  $D_n \subset \mathbb{R}^n$  be the closed unit ball and consider

- $S^3 \times D_3 \subset S^3 \times \mathbb{R}^3 \cong \Psi^{-1} \tilde{U}$ ,
- $D_4 \times S^2 \subset \mathbb{R}^4 \times S^2 \cong \Phi^{-1} \hat{U}$ .

Then  $\tilde{D} := \Psi^{-1}(S^3 \times D_3)$  and  $\hat{D} := \Phi^{-1}(D_4 \times S^2)$  are compact tubular neighborhoods of the vanishing cycle  $\tilde{S} \subset \tilde{U}$  and of the exceptional cycle  $\mathbb{P}^1_{\mathbb{C}} \subset \hat{U}$ , respectively.

Consider the standard diffeomorphism

$$\alpha' : (\mathbb{R}^4 \setminus \{0\}) \times S^2 \xrightarrow{\cong} S^3 \times (\mathbb{R}^3 \setminus \{0\}), \quad (u, v) \mapsto \left( \frac{u}{|u|}, |u|v \right)$$

and restrict it to  $D_4 \times S^2$ . Since

$$\partial(D_4 \times S^2) = S^3 \times S^2 = \partial(S^3 \times D_3)$$

observe that  $\alpha'|_{\partial(D_4 \times S^2)} = \text{id}|_{S^3 \times S^2}$ . Hence  $\alpha'$  induces a standard surgery from  $\mathbb{R}^4 \times S^2$  to  $S^3 \times \mathbb{R}^3$ .

Then  $\tilde{U}$  can be obtained from  $\hat{U}$  by removing  $\hat{D}$  and pasting in  $\tilde{D}$ , by means of the diffeomorphism  $\alpha := \Psi^{-1} \circ \alpha' \circ \Phi$ .

**Proof.** The situation is described by the following commutative diagram

$$\begin{CD} \hat{U} \setminus \mathbb{P}^1_{\mathbb{C}} @>\alpha>> \tilde{U} \setminus \tilde{S} \\ @V\cong \Phi = \varphi VV @VV\cong \Psi V \\ (\mathbb{R}^4 \setminus \{0\}) \times S^2 @>\alpha'>> S^3 \times (\mathbb{R}^3 \setminus \{0\}) \end{CD}$$

which implies that  $\alpha$  induces a diffeomorphism from  $\partial(\tilde{D})$  to  $\partial(\hat{D})$ . The claim follows immediately.  $\square$

3. Global geometry and topology of a conifold transition

Let  $T(Y, \bar{Y}, \tilde{Y})$  be a conifold transition. Then, by definition and the local analysis of the previous section we know that:

- $\text{Sing}(\bar{Y}) = \{p_1, \dots, p_N\}$ , where  $p_i$  is a node;
- there exists a simultaneous resolution  $\phi : Y \rightarrow \bar{Y}$  which is a birational morphism contracting  $N$  rational curves  $E_1, \dots, E_N$ ;
- $\tilde{Y}$  admits  $N$  vanishing cycles  $S_1, \dots, S_N$  which are three-spheres.

Two natural questions then arise:

- (1) Are the homology classes  $[E_1], \dots, [E_N] \in H_2(Y, \mathbb{Z})$  linearly independent? Which is: are the exceptional curves of  $\phi$  homologically independent?
- (2) Same question about  $[S_1], \dots, [S_N] \in H_3(\tilde{Y}, \mathbb{Z})$ , i.e. are the vanishing cycles homologically independent?

The answer is *no* to both questions!

**Example 3.1.** Consider the example given in 1.3 of the conifold in  $\mathbb{P}^4$ . Then  $\tilde{Y} = \{x_3g + x_4h = 0\} \subset \mathbb{P}^4$ ,  $N = 16$ , the resolution  $Y$  contains 16 exceptional rational curves and the smoothing  $\tilde{Y}$  contains 16 vanishing spheres.

For question (1) notice that if  $[E_1], \dots, [E_{16}]$  would be independent then we would have

$$b_2(Y) = b_2(\tilde{Y}) + 16$$

which is clearly contradicting (6).

On the other hand, for question (2) let us compare  $b_3(Y)$  and  $b_3(\tilde{Y})$ .

**Claim 3.2.**  $b_3(Y) = 174, b_3(\tilde{Y}) = 204$ ; then  $b_3(\tilde{Y}) - b_3(Y) = 30$ .

**Proof.** In physics literature, this proof is often realized by invoking the local smoothness of the complex moduli space of a Calabi–Yau three-fold  $Y$ , hence the Bogomolov–Tian–Todorov theorem (see [14,71,73,58]; see also Section 6.1.1). Then it is well defined a tangent space, to such a moduli space, canonically identified with  $H^1(\mathcal{T}_Y)$ , via the Kodaira–Spencer map. The Calabi–Yau condition gives then

$$b_3(Y) = 2 + 2h^{2,1}(Y) = 2 + 2h^1(\mathcal{T}_Y)$$

The statement follows, for both  $Y$  and  $\tilde{Y}$ , by counting their moduli (see [34]).

Actually proving the claim do not need local smoothness of moduli spaces, which is a very deeper concept. In the following we present a more (standard) elementary proof. Although computationally more intricate than the previous one, the following method has the advantage to apply to more general situations: in fact it is not easy to count moduli of a general Calabi–Yau three-fold, even in the case of a complete intersection.

Let start to consider  $\tilde{Y}$  which is the easiest case of a projective hypersurface. In this case there are many methods to compute  $h^{2,1}(\tilde{Y})$ : e.g. it is possible to compute directly  $h^1(\mathcal{T}_{\tilde{Y}})$  by Poincaré residues (see [35]) and to end up by using Calabi–Yau condition. Here is the most elementary procedure to compute  $h^1(\mathcal{T}_{\tilde{Y}})$ .

Since  $\mathcal{N}_{\tilde{Y}|\mathbb{P}^4} \cong \mathcal{O}_{\mathbb{P}^4}(5) \otimes \mathcal{O}_{\tilde{Y}} =: \mathcal{O}_{\tilde{Y}}(5)$ , the tangent sheaf exact sequence gives

$$0 \rightarrow \mathcal{T}_{\tilde{Y}} \rightarrow \mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\tilde{Y}}(5) \rightarrow 0$$

and the associated cohomology long exact sequence starts as follows:

$$0 \rightarrow H^0(\mathcal{T}_{\tilde{Y}}) \rightarrow H^0(\mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}}) \rightarrow H^0(\mathcal{O}_{\tilde{Y}}(5)) \rightarrow H^1(\mathcal{T}_{\tilde{Y}}) \rightarrow H^1(\mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}}) \rightarrow \quad (19)$$

All needed information can then be deduced by the cohomology associated with the Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \rightarrow \mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}} \rightarrow 0 \quad (20)$$

and with the following tensor product, by  $\mathcal{O}_{\tilde{Y}}(5)$ , of the structure sheaf exact sequence of  $\tilde{Y} \subset \mathbb{P}^4$

$$0 \rightarrow \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\mathbb{P}^4}(5) \rightarrow \mathcal{O}_{\tilde{Y}}(5) \rightarrow 0 \tag{21}$$

In fact (20) gives

$$\begin{aligned} 0 \rightarrow \mathbb{C} \rightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5} \rightarrow H^0(\mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}}) \rightarrow H^1(\mathcal{O}_{\tilde{Y}}) \\ \rightarrow H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5} \rightarrow H^1(\mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}}) \rightarrow H^2(\mathcal{O}_{\tilde{Y}}) \rightarrow \dots \end{aligned}$$

Bott formulas

$$h^q(\Omega_{\mathbb{P}^n}^p(a)) = \begin{cases} \binom{a+n-p}{a} \binom{a-1}{p} & \text{for } q = 0, 0 \leq p \leq n \text{ and } a > p \\ 1 & \text{for } 0 \leq p = q \leq n \text{ and } a = 0 \\ \binom{-a+p}{-a} \binom{-a-1}{n-p} & \text{for } q = n, 0 \leq p \leq n \text{ and } a < p - n \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

and the Calabi–Yau condition  $h^1(\mathcal{O}_{\tilde{Y}}) = h^2(\mathcal{O}_{\tilde{Y}}) = 0$ , allow then to conclude that

$$h^0(\mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}}) = 25 - 1 = 24, \quad h^1(\mathcal{T}_{\mathbb{P}^4} \otimes \mathcal{O}_{\tilde{Y}}) = 0$$

On the other hand the cohomology of (21) gives

$$0 \rightarrow \mathbb{C} \rightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \rightarrow H^0(\mathcal{O}_{\tilde{Y}}(5)) \rightarrow H^1(\mathcal{O}_{\tilde{Y}}) \rightarrow \dots$$

Again Bott formulas (22) and Calabi–Yau condition imply that

$$h^0(\mathcal{O}_{\tilde{Y}}(5)) = 126 - 1 = 125$$

Since  $h^0(\mathcal{T}_{\tilde{Y}}) = h^0(\Omega_{\tilde{Y}}^2) = 0$ , the sequence (19) gives

$$h^1(\mathcal{T}_{\tilde{Y}}) = 125 - 24 = 101$$

The previous argument do not apply to the resolution  $Y$ , since it is the complete intersection given by the bi-homogeneous equations (5) in  $\mathbb{P}^1 \times \mathbb{P}^4 =: \mathbb{P}$ . In this case there is no more an Euler sequence like (20), then it is better to directly compute  $h^1(\Omega_{\tilde{Y}}^2)$ . At this purpose dualize the tangent sheaf sequence to get

$$0 \rightarrow \mathcal{N}_{Y|\mathbb{P}}^* \rightarrow \Omega_{\mathbb{P}} \otimes \mathcal{O}_Y \rightarrow \Omega_Y \rightarrow 0 \tag{23}$$

where  $\mathcal{N}_{Y|\mathbb{P}}^* := \mathcal{H}om(\mathcal{N}_{Y|\mathbb{P}}, \mathcal{O}_Y) = \mathcal{I}_Y/\mathcal{I}_Y^2$ , being  $\mathcal{I}_Y$  the ideal sheaf of  $Y \subset \mathbb{P}$ . Then

$$\mathcal{N}_{Y|\mathbb{P}}^* \cong [\mathcal{O}_{\mathbb{P}}(-1, -1) \oplus \mathcal{O}_{\mathbb{P}}(-1, -4)] \otimes \mathcal{O}_Y =: \mathcal{O}_Y(-1, -1) \oplus \mathcal{O}_Y(-1, -4)$$

Since  $Y$  is Calabi–Yau, its canonical sheaf is trivial and the fourth exterior power of (23) gives the following exact sequence

$$0 \rightarrow \mathcal{O}_Y(-2, -5) \otimes \Omega_{\mathbb{P}}^2 \rightarrow \Omega_{\mathbb{P}}^4 \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y(-1, -1) \oplus \mathcal{O}_Y(-1, -4) \rightarrow 0$$

This sequence, tensored by  $\mathcal{O}_{\mathbb{P}}(2, 5)$ , gives then rise to the following one

$$0 \rightarrow \Omega_{\mathbb{P}}^2 \rightarrow \Omega_{\mathbb{P}}^4(2, 5) \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y(1, 4) \oplus \mathcal{O}_Y(1, 1) \rightarrow 0 \tag{24}$$

from which it is possible to compute  $h^1(\Omega_Y^2)$  by passing to the associated long exact sequence in cohomology. In fact, recalling the Calabi–Yau condition for  $Y$ , it follows that

$$\begin{aligned} 0 \rightarrow H^0(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_Y) &\rightarrow H^0(\mathcal{O}_Y(1, 4)) \oplus H^0(\mathcal{O}_Y(1, 1)) \\ &\rightarrow H^1(\Omega_Y^2) \rightarrow H^1(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_Y) \rightarrow \dots \end{aligned} \tag{25}$$

All needed information can then be obtained by suitable twists of the following structure sheaves exact sequences of  $Y \subset \hat{\mathbb{P}} \subset \mathbb{P}$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-1, -1) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow 0 \tag{26}$$

$$0 \rightarrow \mathcal{O}_{\hat{\mathbb{P}}}(-1, -4) \rightarrow \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow \mathcal{O}_Y \rightarrow 0 \tag{27}$$

where  $\hat{\mathbb{P}}$  is the blow up of  $\mathbb{P}^4$  along the plane  $x_3 = x_4 = 0$ , whose equation in  $\mathbb{P}$  is the former in (5), and  $\mathcal{O}_{\hat{\mathbb{P}}}(-1, -4) := \mathcal{O}_{\mathbb{P}}(-1, -4) \otimes \mathcal{O}_{\hat{\mathbb{P}}}$ .

In fact the tensor product of (26) and (27) by  $\Omega_{\mathbb{P}^4}^4(2, 5)$  gives

$$0 \rightarrow \Omega_{\mathbb{P}^4}^4(1, 4) \rightarrow \Omega_{\mathbb{P}^4}^4(2, 5) \rightarrow \Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow 0 \tag{28}$$

$$0 \rightarrow \Omega_{\mathbb{P}^4}^4(1, 1) \otimes \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow \Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow \Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_Y \rightarrow 0 \tag{29}$$

The following Künneth formulas

$$h^v(\Omega_{\mathbb{P}^4}^u(a, b)) = \bigoplus_{\substack{p+r=u \\ q+s=v}} [h^q(\Omega_{\mathbb{P}^1}^p(a)) \cdot h^s(\Omega_{\mathbb{P}^4}^r(b))] \tag{30}$$

and (22) applied to the cohomology long exact sequence of (28) give

$$h^0(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_{\hat{\mathbb{P}}}) = h^0(\Omega_{\mathbb{P}^4}^4(2, 5)) - h^0(\Omega_{\mathbb{P}^4}^4(1, 4)) = 27, \quad h^1(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_{\hat{\mathbb{P}}}) = 0$$

Moreover the tensor product of (26) by  $\Omega_{\mathbb{P}^4}^4(1, 1)$  gives

$$0 \rightarrow \Omega_{\mathbb{P}^4}^4 \rightarrow \Omega_{\mathbb{P}^4}^4(1, 1) \rightarrow \Omega_{\mathbb{P}^4}^4(1, 1) \otimes \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow 0 \tag{31}$$

whose cohomology attains the following results

$$h^0(\Omega_{\mathbb{P}^4}^4(1, 1) \otimes \mathcal{O}_{\hat{\mathbb{P}}}) = h^1(\Omega_{\mathbb{P}^4}^4(1, 1) \otimes \mathcal{O}_{\hat{\mathbb{P}}}) = 0$$

Therefore the cohomology of (29) allows to conclude that

$$\begin{aligned} h^0(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_Y) &= h^0(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_{\hat{\mathbb{P}}}) = 27, \\ h^1(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_Y) &= h^1(\Omega_{\mathbb{P}^4}^4(2, 5) \otimes \mathcal{O}_{\hat{\mathbb{P}}}) = 0 \end{aligned} \tag{32}$$

To compute  $h^0(\mathcal{O}_Y(1, 4))$ , consider the tensor product of (26) and (27) by  $\mathcal{O}_{\mathbb{P}}(1, 4)$ :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(0, 3) \rightarrow \mathcal{O}_{\mathbb{P}}(1, 4) \rightarrow \mathcal{O}_{\hat{\mathbb{P}}}(1, 4) \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_{\hat{\mathbb{P}}} \rightarrow \mathcal{O}_{\hat{\mathbb{P}}}(1, 4) \rightarrow \mathcal{O}_Y(1, 4) \rightarrow 0$$

Again formulas (22) and (30) applied to the cohomology of the first sequence give

$$h^0(\mathcal{O}_{\hat{\mathbb{P}}}(1, 4)) = h^0(\mathcal{O}_{\mathbb{P}}(1, 4)) - h^0(\mathcal{O}_{\mathbb{P}}(0, 3)) = 140 - 35 = 105$$

The cohomology of the second sequence allows to conclude

$$h^0(\mathcal{O}_Y(1, 4)) = 104 \tag{33}$$

Analogously for  $h^0(\mathcal{O}_Y(1, 1))$  one has

$$0 \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathbb{P}}(1, 1) \rightarrow \mathcal{O}_{\hat{\mathbb{P}}}(1, 1) \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_{\hat{\mathbb{P}}}(0, -3) \rightarrow \mathcal{O}_{\hat{\mathbb{P}}}(1, 1) \rightarrow \mathcal{O}_Y(1, 1) \rightarrow 0$$

Then

$$h^0(\mathcal{O}_{\hat{\mathbb{P}}}(1, 1)) = h^0(\mathcal{O}_{\mathbb{P}}(1, 1)) - h^0(\mathcal{O}_{\mathbb{P}}) = 10 - 1 = 9$$

and finally

$$h^0(\mathcal{O}_Y(1, 1)) = 9 \tag{34}$$

Therefore, recalling (25), results (32)–(34) end up the proof giving

$$h^1(\Omega_Y^2) = (104 + 9) - 27 = 86 \quad \square$$

Actually the numbers of nodes in  $\bar{Y}$ , of maximally independent exceptional rational curves in  $Y$  and of maximally independent vanishing cycles in  $\tilde{Y}$  turn out to be deeply related. This fact characterizes the global change in topology induced by a conifold transition, as explained in the following theorem.

**Theorem 3.3** (Refs. [21,62,75,72,55,52]). *Let  $T(Y, \bar{Y}, \tilde{Y})$  be a conifold transition and let*

- $N$  be the number of nodes composing  $\text{Sing}(\bar{Y})$ ,
- $k$  be the maximal number of homologically independent exceptional rational curves in  $Y$ ,
- $c$  be the maximal number of homologically independent vanishing cycles in  $\tilde{Y}$ .

Then:

- (1)  $|\text{Sing}(\bar{Y})| =: N = k + c$ ;
- (2) (Betti numbers)  $b_i(Y) = b_i(\bar{Y}) = b_i(\tilde{Y})$  for  $i \neq 2, 3, 4$ , and

$$\begin{array}{rcccl} b_2(Y) & = & b_2(\bar{Y}) + k & = & b_2(\tilde{Y}) + k \\ & \parallel & & & \parallel \\ b_4(Y) & = & b_4(\bar{Y}) & = & b_4(\tilde{Y}) + k \\ b_3(Y) & = & b_3(\bar{Y}) - c & = & b_3(\tilde{Y}) - 2c \end{array}$$

where vertical equalities are given by Poincaré duality;

- (3) (Hodge numbers)

$$h^{2,1}(\tilde{Y}) = h^{2,1}(Y) + c, \quad h^{1,1}(\tilde{Y}) = h^{1,1}(Y) - k$$

**Remark 3.4.** Note that point (2) of the previous statement implies that the conifold  $\bar{Y}$  do not satisfy Poincaré duality. The difference  $b_4(\bar{Y}) - b_2(\bar{Y}) = k$  is called *the defect of  $\bar{Y}$*  [55].

**Remark 3.5.** Point (3) in Theorem 3.3 has the following geometric interpretation: a conifold transition increases complex moduli by the maximal number of homologically independent vanishing cycles and decreases Kähler moduli by the maximal number of homologically independent exceptional rational curves.

The reader is referred to Section 6.1.1 for a deeper understanding, where the *Calabi–Yau moduli space’s* structure will be quickly described.

**Proof of Theorem 3.3.** Let us denote:

- $P := \text{Sing}(\bar{Y}) = \{p_1, \dots, p_N\}$ , the singular locus of  $\bar{Y}$ ;
- $E := \bigcup_{i=1}^N E_i$ , the exceptional locus of  $Y$ ;
- $S := \bigcup_{i=1}^N S_i$ , the vanishing locus of  $\tilde{Y}$ .

The birational contraction  $\phi : Y \rightarrow \bar{Y}$  induces the isomorphism

$$\phi : Y \setminus E \xrightarrow{\cong} \bar{Y} \setminus P \tag{35}$$

On the other hand, for any  $i = 1, \dots, N$ , by Proposition 2.10 we can construct compact tubular neighborhoods  $\tilde{D}_i$  of the vanishing cycle  $S_i$  in  $\tilde{Y}$  and  $\hat{D}_i$  of the exceptional rational curve  $E_i$  in  $Y$  and diffeomorphisms

$$\alpha_i : \hat{D}_i \setminus E_i \xrightarrow{\cong} \tilde{D}_i \setminus S_i$$

Since we can clearly assume that  $\tilde{D}_i$ 's are all disjoint neighborhoods and the same for  $\hat{D}_i$ 's, the composed morphisms  $\phi \circ \alpha_i^{-1}$  give diffeomorphisms

$$\phi \circ \alpha_i^{-1} : \tilde{D}_i \setminus S_i \xrightarrow{\cong} \bar{D}_i \setminus \{p_i\} \tag{36}$$

where  $\bar{D}_i := \phi(\hat{D}_i)$ . Set:

$$\tilde{D} = \bigcup_{i=0}^N \tilde{D}_i, \quad \bar{D} = \bigcup_{i=0}^N \bar{D}_i$$

By the Ehresmann fibration theorem there exists a diffeomorphism

$$\tilde{Y} \setminus \tilde{D} \xrightarrow{\cong} \bar{Y} \setminus \bar{D}$$

allowing to extend diffeomorphisms (36) to the following global one

$$\psi : \tilde{Y} \setminus S \xrightarrow{\cong} \bar{Y} \setminus P \tag{37}$$

- *Step I.*  $\forall i \neq 2, 3 \ b_i(Y) = b_i(\bar{Y})$  and

$$b_2(Y) = b_2(\bar{Y}) + k \Leftrightarrow b_3(\bar{Y}) = b_3(Y) + N - k.$$

Let  $T(\hat{U}_i, \bar{U}_i, \tilde{U}_i)$  be the local conifold transition (notation as in Section 2) induced by  $T(Y, \bar{Y}, \tilde{Y})$  around the node  $p_i \in P$  and denote:

- $\hat{U} := \bigcup_{i=1}^N \hat{U}_i \subset Y, Y^* := Y \setminus E, \hat{U}^* := \hat{U} \setminus E,$
- $\bar{U} := \bigcup_{i=1}^N \bar{U}_i \subset \bar{Y}, \bar{Y}^* := \bar{Y} \setminus P, \bar{U}^* := \bar{U} \setminus P.$

Then:

- $\hat{U}^* = Y^* \cap \hat{U}$  and  $Y = Y^* \cup \hat{U},$
- $\bar{U}^* = \bar{Y}^* \cap \bar{U}$  and  $\bar{Y} = \bar{Y}^* \cup \bar{U},$

and we are in a position to apply Mayer–Vietoris machinery to the couples  $(Y^*, \hat{U})$  and  $(\bar{Y}^*, \bar{U})$  to get the following two long exact sequences in homology

$$\dots \rightarrow H_i(\hat{U}^*) \rightarrow H_i(Y^*) \oplus H_i(\hat{U}) \rightarrow H_i(Y) \rightarrow H_{i-1}(\hat{U}^*) \rightarrow \dots \tag{38}$$

$$\dots \rightarrow H_i(\bar{U}^*) \rightarrow H_i(\bar{Y}^*) \oplus H_i(\bar{U}) \rightarrow H_i(\bar{Y}) \rightarrow H_{i-1}(\bar{U}^*) \rightarrow \dots \tag{39}$$



By straight line homotopy we have

$$H_i(\hat{U}) \cong H_i(E) \cong \begin{cases} \mathbb{Z}^N & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

as a consequence of Proposition 2.2 and

$$H_i(\bar{U}) \cong H_i(P) \cong \begin{cases} \mathbb{Z}^N & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

as a consequence of Proposition 2.1. The diffeomorphism  $\phi$ , given in (35), induces then the following isomorphisms in homology

$$H_i(\hat{U}^*) \cong H_i(\bar{U}^*) \cong \bigoplus_{i=1}^N H_i(S^3 \times S^2) \cong \begin{cases} \mathbb{Z}^N & \text{if } i = 0, 2, 3, 5 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

and

$$H_i(Y^*) \cong H_i(\bar{Y}^*) \tag{43}$$

Introduce isomorphisms (40)–(43), as vertical arrows connecting sequences (38) and (39). The Steenrod five-lemma gives then

$$\forall i \neq 2, 3, \quad b_i(Y) = b_i(\bar{Y}) \tag{44}$$

Moreover gluing the two sequences by identifying the isomorphic poles, they reduce to the following diagram

$$\begin{array}{ccccccc}
 (45) & 0 & \longrightarrow & H_4(Y^*) & \longrightarrow & H_4(Y) & \longrightarrow & \dots \\
 & & & \nearrow & & \searrow & & \\
 & & & H_3(Y) & & H_2(Y^*) \oplus \mathbb{Z}^N & \longrightarrow & H_2(Y) \\
 & \nearrow & & \searrow & \nearrow & \searrow & & \\
 \dots & H_3(Y^*) & & & \mathbb{Z}^N & & & 0 \\
 & \searrow & & \nearrow & \searrow & \nearrow & & \\
 & & & H_3(\bar{Y}) & & H_2(\bar{Y}^*) & \longrightarrow & H_2(\bar{Y})
 \end{array} \tag{45}$$

Then we get the following relations on Betti numbers

$$b_4(Y^*) - b_4(Y) + b_3(Y^*) - b_3(Y) + N - (b_2(Y^*) + N) + b_2(Y) = 0,$$

$$b_4(Y^*) - b_4(Y) + b_3(Y^*) - b_3(\bar{Y}) + N - b_2(\bar{Y}^*) + b_2(\bar{Y}) = 0$$

and their difference gives

$$b_2(Y) - b_2(\bar{Y}) = b_3(Y) - b_3(\bar{Y}) + N$$

- Step II.  $\forall i \neq 3, 4 \quad b_i(\tilde{Y}) = b_i(\bar{Y})$  and

$$b_3(\tilde{Y}) = b_3(\bar{Y}) + c \Leftrightarrow b_4(\tilde{Y}) = b_4(\bar{Y}) + N - c$$

Let  $T(\hat{U}_i, \bar{U}_i, \tilde{U}_i)$  be the local conifold induced near the node  $p_i \in P$ , as before. Let us denote

- $\tilde{U} := \bigcup_{i=1}^N \tilde{U}_i \subset \tilde{Y}$ ,  $\tilde{Y}^* := \tilde{Y} \setminus S$ ,  $\tilde{U}^* := \tilde{U} \setminus S$ .

Then

$$\circ \tilde{U}^* = \tilde{Y}^* \cap \tilde{U} \text{ and } \tilde{Y} = \tilde{Y}^* \cup \tilde{U}$$

and Mayer–Vietoris sequence for the couple  $(\tilde{Y}^*, \tilde{U})$  gives

$$\dots \rightarrow H_i(\tilde{U}^*) \rightarrow H_i(\tilde{Y}^*) \oplus H_i(\tilde{U}) \rightarrow H_i(\tilde{Y}) \rightarrow H_{i-1}(\tilde{U}^*) \rightarrow \dots \tag{46}$$

Proposition 2.5 and straight line homotopy give

$$H_i(\tilde{U}) \cong H_i(S) \cong \bigoplus_{i=1}^N H_i(S^3) \cong \begin{cases} \mathbb{Z}^N & \text{if } i = 0, 3 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

Moreover the diffeomorphism  $\psi$  given in (37) induces the following isomorphisms in homology

$$H_i(\tilde{U}^*) \cong H_i(\bar{U}^*) \cong \bigoplus_{i=1}^N H_i(S^3 \times S^2) \cong \begin{cases} \mathbb{Z}^N & \text{if } i = 0, 2, 3, 5 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

and

$$H_i(\tilde{Y}^*) \cong H_i(\bar{Y}^*) \tag{49}$$

As before, apply the Steenrod five-lemma to conclude that

$$\forall i \neq 3, 4 \quad b_i(\tilde{Y}) = b_i(\bar{Y}) \tag{50}$$

and glue sequences (39) and (46) to get the following diagram

$$\begin{array}{ccccccc} & & H_4(\tilde{Y}) & & H_3(\tilde{Y}^*) \oplus \mathbb{Z}^N & \longrightarrow & H_3(\tilde{Y}) \cdots \\ & & \nearrow & & \nearrow & & \\ 0 & \longrightarrow & H_4(\tilde{Y}^*) & & \mathbb{Z}^N & & \\ & & \searrow & & \searrow & & \\ & & H_4(\bar{Y}) & & H_3(\bar{Y}^*) & \longrightarrow & H_3(\bar{Y}) \cdots \\ & & & & & & \\ \dots & & & & & & \\ & & & & \mathbb{Z}^N & \longrightarrow & H_2(\tilde{Y}^*) \longrightarrow H_2(\tilde{Y}) \longrightarrow 0 \\ & & & & \nearrow & & \\ \dots & & & & & & \end{array} \tag{51}$$

Then we get the following relations on Betti numbers

$$b_4(\tilde{Y}^*) - b_4(\tilde{Y}) + N - (b_3(\tilde{Y}^*) + N) + b_3(\tilde{Y}) - N + b_2(\tilde{Y}^*) - b_2(\tilde{Y}) = 0,$$

$$b_4(\tilde{Y}^*) - b_4(\tilde{Y}) + N - b_3(\tilde{Y}^*) + b_3(\tilde{Y}) - N + b_2(\tilde{Y}^*) - b_2(\tilde{Y}) = 0$$

and their difference gives

$$b_3(\tilde{Y}) - b_3(\bar{Y}) = b_4(\tilde{Y}) - b_4(\bar{Y}) + N$$

- Step III. Let  $k$  and  $c$  be the same parameters defined in Steps I and II, respectively. Then

$$|\text{Sing}(\tilde{Y})| =: N = k + c := b_2(Y) - b_2(\bar{Y}) + b_3(\tilde{Y}) - b_3(\bar{Y})$$



Set

$$I := \text{Im}[\kappa : \mathbb{Z}^N = H_2(E) \rightarrow H_2(Y)]$$

Then  $k := \text{rk}(I)$  is the number of linear independent classes of exceptional curves in  $H_2(Y)$ . Since

$$0 \rightarrow I \rightarrow H_2(Y) \rightarrow H_2(\tilde{Y}) \rightarrow 0$$

is a short exact sequence, it follows that

$$b_2(Y) = b_2(\tilde{Y}) + k$$

On the other hand set

$$K := \text{ker}[\gamma : \mathbb{Z}^N \cong H_3(S) \rightarrow H_3(\tilde{Y})]$$

Then  $N - c := \text{rk}(K)$  is the number of linear independent relations on the classes of vanishing cycles in  $H_3(\tilde{Y})$ . Since

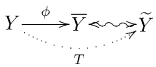
$$0 \rightarrow H_4(\tilde{Y}) \rightarrow H_4(Y) \rightarrow K \rightarrow 0$$

is a short exact sequence, it follows that

$$b_4(Y) = b_4(\tilde{Y}) + N - c \quad \square$$

### 3.1. What about more general geometric transitions?

The local and global topology and geometry of a general geometric transition



can actually be very intricate, depending on the nature of  $\text{Sing}(\tilde{Y})$  and on the geometry of the exceptional locus of  $\phi$ . For this reason no general results similar to [Proposition 2.10](#) and [Theorem 3.3](#) are known. Anyway, under some (strong) condition on  $\text{Sing}(\tilde{Y})$ , somewhat can be said.

First of all let us assume that  $\text{Sing}(\tilde{Y}) = \{p_1, \dots, p_r\}$  is composed only by isolated hypersurface singularity.

In this case, given a one-parameter flat smoothing  $\mathcal{Y} \rightarrow \Delta^1$  of the singular point  $p_i$ , the local topology of  $\mathcal{Y}$  near  $p_i$  is explained by the Milnor’s analysis [49]. Call  $B$  the union of all of the Milnor’s fibres  $B_{p_i}$ , which have the homology type of a bouquet of three-spheres. Interpolate the relative homology long exact sequences of  $(\tilde{Y}, \text{Sing}(\tilde{Y}))$  and  $(\tilde{Y}, B)$ , like in step IV of the proof of [Theorem 3.3](#), to get the first part of the following theorem.

**Theorem 3.6** ([55, Theorem (3.2)]). *Let  $\tilde{Y}$  be a normal projective three-fold with only isolated hypersurface singularities, admitting a smoothing  $\tilde{Y}$ . For any  $p \in \text{Sing}(\tilde{Y})$  call  $m(p) := h_3(B_p)$  the Milnor number of  $p$ . Then the defect of  $\tilde{Y}$  is related to Milnor numbers as follows:*

$$k := b_4(\tilde{Y}) - b_2(\tilde{Y}) = b_3(\tilde{Y}) + \sum_{p \in \text{Sing}(\tilde{Y})} m(p) - b_3(\tilde{Y}) \tag{56}$$

Moreover if all of the singularities of  $\tilde{Y}$  are rational then

$$W(\tilde{Y})/C(\tilde{Y}) := \langle \text{Weil divisors of } \tilde{Y} \rangle_{\mathbb{Z}} / \langle \text{Cartan divisors of } \tilde{Y} \rangle_{\mathbb{Z}}$$

is a finitely generated abelian group. In particular if  $h^2(\mathcal{O}_{\bar{Y}}) = 0$  then

$$k = \text{rk}(W(\bar{Y})/C(\bar{Y}))$$

giving a further interpretation of the defect of  $\bar{Y}$ .

Since the Milnor fibre of a node  $p$  has the homology type of a single three-sphere,  $m(p) = 1$  and (56) gives (1) and the right part of formulas (2) in Theorem 3.3.

The last part of the previous statement is proved by employing results of Dimca [24] and Steenbrink [67].

Moreover Theorem 3.6, joint with results of Reid [60], allows to generalize Theorem 3.3 to the case of a geometric transition whose birational contraction is a small one, as follows.

**Theorem 3.7** ([55, Example (3.8)]). *Let  $T(Y, \bar{Y}, \tilde{Y})$  be a geometric transition whose birational contraction  $\phi : Y \rightarrow \bar{Y}$  is a composition of type I primitive contractions. Then  $\text{Sing}\bar{Y} = \{p_1, \dots, p_r\}$ , where  $p_i$  is an isolated, rational singularity. Let  $C_i := \phi^{-1}(p_i)$  and  $n_i$  be the number of irreducible components of  $C_i$ . If  $k$  is the rank of the free abelian group generated in  $H_2(Y)$  by the homology classes of  $C_1, \dots, C_r$ , then*

$$b_2(\tilde{Y}) = b_2(Y) - k, \quad b_3(\tilde{Y}) = b_3(Y) + \sum_{i=1}^r n_i + \sum_{i=1}^r m(p_i) - 2k$$

As far as I know, dropping assumptions on  $\text{Sing}(\bar{Y})$  leads to no more than interesting conjectures and examples. The interested reader is referred to [52, Section 3 and Appendix A], for some geometric and physical interpretation of parameters  $N, k, c$  for more general transitions, and to [41] for a computation of these parameters in examples of transitions whose  $\bar{Y}$  admits non-isolated singularities (see also Section 6.2).

#### 4. Classification of geometric transitions

By definition, a general geometric (not necessarily conifold) transition  $T(Y, \bar{Y}, \tilde{Y})$  is always associated with a birational contraction of a Calabi–Yau three-fold  $Y$  to a normal variety  $\bar{Y}$ . Then the ingredients of a classification are the following:

- (1) to classify the birational contractions  $\phi : Y \rightarrow \bar{Y}$  which may occur,
- (2) among them, to select those admitting a smoothable target  $\bar{Y}$ .

Let us start with the first point of our program.

##### 4.1. A little bit of Mori theory for Calabi–Yau three-folds

Let  $Y$  be a Calabi–Yau three-fold and consider the Picard group

$$\begin{aligned} \text{Pic}(Y) &:= \langle \text{invertible sheaves} \rangle_{\mathbb{Z}} / \text{isomorphism} (\cong), \\ &\cong \langle \text{divisors} \rangle_{\mathbb{Z}} / \text{linear equivalence} (\equiv) \end{aligned}$$

**Remark 4.1.** There is a canonical isomorphism

$$\text{Pic}(Y) \cong H^2(Y, \mathbb{Z}) \tag{57}$$

In fact, since  $Y$  is smooth, it is well known that  $\text{Pic}(Y) \cong H^1(Y, \mathcal{O}_Y^*)$ . The long exact cohomology sequence associated with the *exponential sequence*

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\text{exp}} \mathcal{O}^* \rightarrow 1$$

gives the claim as a consequence of the Calabi–Yau condition  $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ .

The *Kleiman space* is the following real vector space

$$H^2(Y, \mathbb{R}) \cong H^2(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^\rho \tag{58}$$

whose dimension is clearly  $\rho = \text{rk}(\text{Pic}(Y))$ , called the *Picard number of  $Y$* .

**Definition 4.2.** A divisor  $D$  of  $Y$  is called *nef (numerically effective)* if any curve  $C$  in  $Y$  intersects  $D$  non-negatively i.e.

$$(D \cdot C) \geq 0$$

**Definition 4.3** (The closed Kähler cone). The *closed Kähler cone*  $\bar{\mathcal{K}}(Y)$  of  $Y$  is the cone generated in the Kleiman space  $H^2(Y, \mathbb{R})$  by the classes of nef divisors.

**Definition 4.4** (The closed Mori cone). The dual construction with respect to the perfect pairing

$$(\cdot) : H^2(Y, \mathbb{R}) \otimes H_2(Y, \mathbb{R}) \rightarrow \mathbb{R}$$

induced by the intersection product, gives rise to the *closed Mori cone*  $\overline{NE}(Y)$ .

**Theorem 4.5** (Kleiman ampleness criterion [43]). *A divisor  $D$  of  $Y$  (not necessarily neither Calabi–Yau nor three-dimensional) is ample if and only if*

$$\forall Z \in \overline{NE}(Y) \setminus \{0\} \quad (D \cdot Z) > 0$$

**Corollary 4.6.** *Let  $Y$  be Calabi–Yau variety. The interior  $\mathcal{K}(Y)$  of  $\bar{\mathcal{K}}(Y)$  is the cone generated by the Kähler classes in the Kleiman space  $H^2(Y, \mathbb{R})$ .*

**Proof.** The criterion 4.5 ensures that  $\mathcal{K}(Y)$  is the cone generated by the classes of ample divisors in  $H^2(Y, \mathbb{R})$ . A divisor is ample if and only if its fundamental form is positive, then  $D$  is ample if and only if  $[D] \in H^2(Y, \mathbb{R})$  is the class of a Kähler form, since the Calabi–Yau condition ensures that  $H^2(Y, \mathbb{C}) \cong H^{1,1}(Y)$ .  $\square$

**Theorem 4.7** (Of the Mori cone [50]). *The negative part of  $\overline{NE}(Y)$  ( $Y$  not necessarily neither Calabi–Yau nor three-dimensional) is rational and polyhedral i.e. there exists a collection  $\{C_i\}_{i \in I}$  of rational curves in  $Y$  such that*

$$\overline{NE}(Y)_- := \overline{NE}(Y) \cap \{Z \in \overline{NE}(Y) \mid (K_Y \cdot Z) < 0\} = \sum_{i \in I} \mathbb{R}_{\geq 0}[C_i].$$

**Theorem 4.8** (Of the Kähler cone [78,79]). *Let  $Y$  be a Calabi–Yau three-fold and consider the cubic cone in  $H^2(Y, \mathbb{R})$  given by the cup-product*

$$W^* := \{[D] \in H^2(Y, \mathbb{R}) \mid D^3 = 0\} \tag{59}$$

(it is the cone projecting a cubic hypersurface  $W \subset \mathbb{P}(H^2(Y, \mathbb{R})) = \mathbb{P}_{\mathbb{R}}^{\rho-1}$ ). Then

$$W^* \cap \bar{\mathcal{K}}(Y) \subset \partial \bar{\mathcal{K}}(Y) \tag{60}$$

and  $\bar{\mathcal{K}}(Y)$  is locally polyhedral away from  $W^*$ . In particular  $\partial \bar{\mathcal{K}}(Y) \setminus W^*$  is composed by codimension 1 faces and their intersections.

**Remark 4.9.** Relation (60) is an immediate consequence of the definition of  $\bar{\mathcal{K}}(Y)$ . In fact if there exists  $[D] \in W^* \cap \mathcal{K}(Y)$  then  $D$  should be ample, implying that  $D^3 > 0$  and contradicting (59).

**Remark 4.10.** By Corollary 4.6,  $\partial\bar{\mathcal{K}}(Y) \cap W^*$  parameterizes all the possible degenerations of a Kähler metric on  $Y$  (see [52, Section 3]).

4.2. Primitive contractions and primitive transitions

**Definition 4.11.** Let  $\phi : Y \rightarrow \bar{Y}$  be a birational contraction of a Calabi–Yau variety to a normal one.  $\phi$  is called *primitive* (or alternatively *extremal*, as explained in Remark 4.15(1)) if it cannot be factored into birational morphisms of normal varieties. Any associated transition  $T(Y, \bar{Y}, \check{Y})$  is called a *primitive* (or *extremal*) transition.

**Proposition 4.12** (Contractions by the Mori–Kähler cones point of view). *There is a correspondence*

$$\{\phi : Y \rightarrow \bar{Y} \text{ contraction from Calabi–Yau to normal}\} \\ \Leftrightarrow (\partial\bar{\mathcal{K}}(Y) \setminus W^*)_{\mathbb{Q}} \Leftrightarrow (\partial\bar{NE}(Y) \cap \bar{NE}(Y)_{-})_{\mathbb{Q}}$$

where  $(\cdot)_{\mathbb{Q}}$  means “rational points of”. In particular

$$\phi \text{ is primitive} \Leftrightarrow \text{it corresponds to a class } [D] \text{ in the interior of a codimension} \\ \text{1 face of } \bar{\mathcal{K}}(Y) \\ \Leftrightarrow \text{it corresponds to a class generating an extremal ray of } \bar{NE}(Y)$$

**Sketch of proof.** Let  $H$  be a hyperplane section of  $\bar{Y}$ . Since  $\bar{Y}$  is normal we can assume

$$H \cap \text{Sing}(\bar{Y}) = \emptyset \tag{61}$$

Look at the pull-back  $\phi^*H$ . The Kleinman criterion 4.5 ensures that

$$\forall Z \in \bar{NE}(Y), \quad (\phi^*H \cdot Z) \geq 0$$

In particular, if  $E$  is the exceptional locus of  $\phi$ , the *projection formula* and (61) give

$$(\phi^*H \cdot Z) = 0 \Leftrightarrow Z \text{ is the class of a curve } C \subset E$$

Then  $(\phi^*H \cdot \cdot)$  defines a hyperplane in  $H_2(Y, \mathbb{R})$  cutting  $\bar{NE}(Y)$  along an extremal face. By duality  $[\phi^*H]$  generates a ray living in a codimension 1 face of the polyhedral part of the Kähler cone i.e.

$$\mathbb{R}_{\geq 0}[\phi^*H] \subset \partial\bar{\mathcal{K}}(Y) \setminus W^*$$

Notice that the contraction  $\phi$  can be factored into birational morphisms if there exists a curve  $C$  in  $E$  and  $Z_1, Z_2 \in \bar{NE}(Y)$  such that

$$\mathbb{R}_{\geq 0}Z_1 \neq \mathbb{R}_{\geq 0}Z_2 \quad \text{and} \quad [C] = Z_1 + Z_2 \tag{62}$$

Hence

$$\phi \text{ is primitive} \Leftrightarrow \forall C \subset E \mathbb{R}_{\geq 0}[C] \text{ is the same extremal ray of } \bar{NE}(Y) \\ \Leftrightarrow \mathbb{R}_{\geq 0}[\phi^*H] \text{ is not on the intersection of two codimension 1 faces of } \bar{\mathcal{K}}(Y) \\ \Leftrightarrow \phi^*H \text{ is an interior point of an extremal codimension 1 face} \quad \square$$

**Corollary 4.13.** *Let  $T(Y, \bar{Y}, \tilde{Y})$  be a geometric transition and  $\phi : Y \rightarrow \bar{Y}$  the associated birational contraction. Then  $\phi$  can always be factored into a composite of a finite number of primitive contractions.*

**Remark 4.14.** The finiteness of the factorization process follows from the fact that any primitive contraction reduces by 1 the Picard number.

**Remark 4.15.** The correspondence given in Proposition 4.12 from contraction morphisms and rational points of the boundary of the Kähler cone is not a 1:1 correspondence. Actually all the rational classes living in the interior of the same codimension 1 face of the  $\bar{\mathcal{K}}(Y)$  correspond to the same primitive birational contraction.

It is then possible to conclude that:

- (1) there is a 1:1 correspondence between primitive contractions and either codimension 1 faces of the Kähler cone  $\bar{\mathcal{K}}(Y)$  or extremal rays of the Mori cone  $\overline{NE}(X)$  ([79, Fact 1]); for this reason primitive contractions (transitions) are also called *extremal* contractions (transitions) [51];
- (2) there is a 1:1 correspondence between codimension  $r$  faces of the Kähler cone  $\bar{\mathcal{K}}(Y)$  and birational contractions from a Calabi–Yau three-fold to a normal variety composed by  $r$  primitive contractions.

**Theorem 4.16** (Classification of primitive contraction [79]). *Let  $\phi : Y \rightarrow \bar{Y}$  be a primitive contraction from a Calabi–Yau three-fold to a normal variety. Then one of the following is true:*

- type I :  $\phi$  is small and the exceptional locus  $E$  is composed of finitely many rational curves;*
- type II :  $\phi$  contracts a divisor down to a point; in this case  $E$  is irreducible and in particular it is a (generalized) del Pezzo surface (see [59]);*
- type III :  $\phi$  contracts a divisor down to a curve  $C$ ; in this case  $E$  is still irreducible and it is a conic bundle over a smooth curve  $C$ .*

**Definition 4.17** (Classification of primitive transitions). A transition  $T(Y, \bar{Y}, \tilde{Y})$  is called of *type I, II or III* if it is *primitive* and if the associated birational contraction  $\phi : Y \rightarrow \bar{Y}$  is of type I, II or III, respectively.

### 4.3. Smoothing the target space $\bar{Y}$

Let us now consider the second point of the classification program given at the beginning of the present section.

Let  $\phi : Y \rightarrow \bar{Y}$  be a birational contraction of a Calabi–Yau three-fold to a normal one. The problem is to select all those contractions admitting a smoothable target space  $\bar{Y}$ .

To answer need to analyze the singularities of  $\bar{Y}$  and actually the geometry of the exceptional locus of  $\phi$ . Since this is a very hard (and almost completely open) problem for a general birational contraction  $\phi$  let us at first restrict to consider the case of primitive contractions, as classified by Theorem 4.16.

#### 4.3.1. Transitions of type I

$\phi$  is the contraction of  $E_1, \dots, E_N$  with  $E_i \cong \mathbb{P}^1$ . Then:

- (1)  $\bar{Y}$  has  $N$  isolated singularities  $p_i = \phi(E_i)$ .



- (2) Reid proved that isolated singularities of this kind are actually *compound Du Val (cDV) singularities* (see [60, Corollary (1.12)]) i.e. they admit local equation of the following type

$$f(x, y, z) + tg(x, y, z, t) = 0 \text{ in } \mathbb{C}^4 \tag{63}$$

where  $f(x, y, z) = 0$  is the local equation in  $\mathbb{C}^3$  of a *rational surface singularity* (also known as *Du Val singularity*, see [59,4]). Eq. (63) actually means that our three-dimensional singularity reduces to a rational surface singularity on a suitable section.

- (3) If  $Y$  is *general* (in its complex moduli space) such a singular point can be reduced to be an ordinary double point (a *node*) i.e.

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{and} \quad g(x, y, z, t) = t$$

This fact follows from the following theorem.

**Theorem 4.18** ([53, Theorem B; 55, Theorem 2.4; 54, Theorem 2.7; 36, Theorem 3.8]). *Let  $Y$  be a Calabi–Yau three-fold and suppose  $\phi : Y \rightarrow \bar{Y}$  is a birational contraction morphism such that  $\bar{Y}$  has isolated, canonical, complete intersection singularities. Then there is a deformation of  $\bar{Y}$  to a variety with at worst ordinary double points.*

**Remark 4.19.** By the previous point (2) we simply may assume  $Y$  to admit isolated cDV singular points which are, in particular, hypersurfaces singularities. Then for the present purpose it suffices Theorem 2.4 of [55] to conclude.

Anyway we preferred to state **Theorem 4.18** in the improved form given by Gross [36, Theorem 3.8] for further applications in the case of more general transitions.

**Remark 4.20.** In [36, Corollary 3.10],  $Y$  may be also assumed to be  $\mathbb{Q}$ -factorial (i.e.  $\text{rk}(W(Y)/C(Y)) = 0$ , see **Theorem 3.6**) with *terminal singularities*. In fact, by results of Namikawa and Steenbrink [55,53], in this case there are small deformations  $\mathcal{Y} \rightarrow \Delta$  and  $\bar{\mathcal{Y}} \rightarrow \Delta$  of  $Y$  and  $\bar{Y}$ , respectively, such that the morphism  $\phi : Y \rightarrow \bar{Y}$  can be deformed to a morphism  $\varphi : \mathcal{Y} \rightarrow \bar{\mathcal{Y}}$ . In particular, for  $t \neq 0$ ,  $\mathcal{Y}_t$  is smooth and  $\bar{\mathcal{Y}}_t$  still has isolated complete intersection singularities but admits a crepant resolution  $\varphi_t : \mathcal{Y}_t \rightarrow \bar{\mathcal{Y}}_t$ . Then one applies **Theorem 4.18** to  $\bar{\mathcal{Y}}_t$ .

- (4) The last step is the following result essentially due to Friedman.

**Theorem 4.21** (27,28,36 Theorem 5.1). *If  $\phi : Y \rightarrow \bar{Y}$  is of type I and  $\bar{Y}$  has at most ordinary double points then  $\bar{Y}$  admits a Calabi–Yau smoothing  $\bar{Y}$  except for the case  $N = 1$  (which is: if  $\phi$  contracts a single  $\mathbb{P}^1$  to a node  $p$  then the germ  $(\bar{Y}, p)$  is rigid).*

**Sketch of proof.** The key fact in proving the previous theorem is that the exceptional curves  $E_1, \dots, E_N$  of  $\phi$  must be *homologically dependent* in  $H_2(Y, \mathbb{Z})$ , since  $\phi$  is a primitive contraction i.e. it is the contraction of a unique extremal ray  $\mathbb{R}_{\geq 0}[E_i] \subset \overline{NE}(Y)$ . Then there is a non-trivial linear dependence relation on  $[E_1], \dots, [E_N]$ , except for  $N = 1$ . Results of Friedman et al. conclude the proof (see [27, Remark 4.5; 28, Proposition 8.7; 72, Theorem 0.1; 54, Theorem 2.5]).  $\square$

*Conclusion:* If  $T(Y, \bar{Y}, \tilde{Y})$  is a type I transition then the exceptional locus  $E$  is composed by  $N \geq 2$  rational curves. Moreover if  $Y$  is general then  $T$  is a conifold transition contracting  $N \geq 2$  rational curves down to nodes.

In particular  $E$  can never be isomorphic to a single  $\mathbb{P}^1$ .

#### 4.3.2. Transitions of type II

$\phi$  is the contraction of an irreducible divisor  $E$  which is a generalized del Pezzo surface. Then:

- (1)  $\bar{Y}$  has one singular point  $p = \phi(E)$ , which is a *canonical singularity* [59,61]. In particular the exceptional surface  $E$  is either a normal del Pezzo surface of degree  $k \leq 9$  or a non-normal del Pezzo surface as classified in [63] and it is rational for  $k \geq 4$ . Moreover  $\phi$  is a (weighted, for  $k \leq 2$ ) blowing up of  $\bar{Y}$  at  $p$ .
- (2)  $k = \deg E$  is the Reid’s invariant of the singularity  $p = \phi(E)$ . In particular we get that (see [59], Proposition 2.9 and Corollary 2.10):
  - $k \leq 2$ : then  $p$  is a hypersurface singularity whose local equation is known,
  - $k \geq 3$ : then  $p$  is a singularity of multiplicity  $k$  and minimal embedding dimension  $\dim(m_p/m_p^2) = k + 1$ .

In particular, for  $k \leq 4$ ,  $p$  is a complete intersection singularity and, on the contrary, for  $k \geq 5$ ,  $p$  is never a complete intersection singularity.

We can then apply Theorem 4.18 to conclude that there exists a smoothing  $\tilde{Y}$  of  $\bar{Y}$  when  $E$  is normal and  $\deg E \leq 4$ , since  $p$  can never be a node.

- (3) Normal rational del Pezzo surfaces  $E$  of degree  $k \geq 5$  are classified in [69]. In particular  $E$  is smoothable, meaning that the embedding  $E \hookrightarrow Y$  can be deformed to an embedding  $E' \hookrightarrow Y'$  where both  $E'$  and  $Y'$  are smooth. When  $E$  is smooth  $p$  is analytically isomorphic to the vertex of a cone over  $E$  [36, Proposition 5.4]. The deformation theory of a cone over a smooth del Pezzo surface of degree  $5 \leq k \leq 9$  is known and precisely:
  - $k = 5$ : then  $p$  is a codimension 3 singularity and there exists a smoothing  $\tilde{Y}$  of  $\bar{Y}$  since locally  $\bar{Y}$  is a Pfaffian subscheme [44],
  - $6 \leq k \leq 9$ : then the considered cones are toric varieties and by [1] we get:
    - $k = 6$ : then there are two distinct smoothings  $\tilde{Y}$  given either by the generic hyperplane section of a cone over  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$  or by two generic hyperplane sections of a cone over  $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ ;
    - $k = 7$ : then there is a smoothing  $\tilde{Y}$  given by the generic hyperplane section of a cone over  $\mathbb{P}^3$  blown up at a point, suitably embedded in  $\mathbb{P}^8$ ;
    - $k = 8$ : then either  $E \cong \mathbb{P}^1 \times \mathbb{P}^1$  and there exists a smoothing  $\tilde{Y}$  given by the generic hyperplane section of a cone over a suitably embedded  $\mathbb{P}^3$ , or  $E$  is the Hirzebruch surface  $F_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$  and  $(\bar{Y}, p)$  is rigid;
    - $k = 9$ : then  $E \cong \mathbb{P}^2$  and  $(\bar{Y}, p)$  is rigid (this case follows also by [64]).
- (4) On the other hand  $E$  is a surface embedded in the smooth three-fold  $Y$ , which means that  $E$  cannot admit non-hypersurface singularities. This fact gives significative constraints on the non-normal case implying that:
  - if  $E$  is non-normal then it is a suitable projection of a Hirzebruch surface  $F_a := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$  having  $E^3 = \deg E = 7$  [36, Theorem 5.2]. In this particular case there exists a smoothing  $\tilde{Y}$  of  $\bar{Y}$  [36, Lemma 5.6].

*Conclusion:* If  $T(Y, \bar{Y}, \tilde{Y})$  is a type II transition then  $Y$  is the blow up of  $\bar{Y}$  at the singular point and the exceptional divisor  $E$  is either a rational, normal, del Pezzo surface of degree  $k \leq 8$  or a non-normal del Pezzo surface of degree 7. In the first case if

- $k \leq 3$ : then  $\bar{Y}$  has a hypersurface singularity,
- $k = 4$ : then  $\bar{Y}$  has a complete intersection singularity,
- $k = 5$ : then  $\bar{Y}$  is locally a Pfaffian subscheme,
- $6 \leq k \leq 8$ : then  $\bar{Y}$  is locally a cone over  $E$  admitting a toric structure.

In particular  $E$  can never be isomorphic to either  $\mathbb{P}^2$  or  $F_1$  [36, Theorem 5.8].

### 4.3.3. Transitions of type III

$\phi$  is the contraction of an irreducible divisor  $E$  down to a smooth curve  $C \subset \bar{Y}$ . Then:

- (1)  $C = \text{Sing}(\bar{Y})$  is a smooth curve of *canonical singularities* of  $\bar{Y}$ ; apply Theorem 2.2 of [59] to conclude that  $C$  is entirely composed of cDV singular points since  $E$  is essentially the only possible exceptional divisor of a crepant resolution of  $\bar{Y}$  and it gives a one-dimensional fibration over  $C$ ;
- (2) the restriction  $\phi|_E : E \rightarrow C$  exhibit  $E$  like a conic bundle over  $C$ , whose fibre is either a smooth conic, a union of two lines meeting at a point, or a double line; in particular if the general fibre is smooth then  $E$  is normal [79, Theorem 2.2], [80];
- (3) let  $\hat{E}$  be the normalization of  $E$  and  $f : \hat{E} \rightarrow Y$  the induced map; saying  $\text{Def}(f)$  the deformations space of  $f$  like in [57] and  $\text{Def}(Y)$  the Kuranishi space of  $Y$ , there is a natural map

$$\text{Def}(f) \rightarrow \text{Def}(Y)$$

then: *the genus of  $C$  is less or equal to the codimension of  $\text{Im}(\text{Def}(f) \rightarrow \text{Def}(Y))$*  [37, Proposition 1.2];

- (4) by the previous step: *if  $g(C) \geq 1$  then there exists a smoothing  $\tilde{Y}$  of  $\bar{Y}$*  [37, Theorem 1.3]; in fact there exists a deformation  $\mathcal{Y} \rightarrow \Delta$  of  $Y$  such that the exceptional divisor  $E$  do not deform to general  $\mathcal{Y}_t, t \in \Delta$  since

$$\text{codim}(\text{Im}(\text{Def}(f) \rightarrow \text{Def}(Y))) \geq 1$$

the contraction  $\phi$  yields a contraction  $\mathcal{Y} \rightarrow \bar{\mathcal{Y}}$ , where  $\bar{\mathcal{Y}} \rightarrow \Delta$  is the deformation induced by  $\mathcal{Y}$  via the natural map  $\text{Def}(Y) \rightarrow \text{Def}(\bar{Y})$ , which exists by [46, Proposition 11.4]; for general  $t \in \Delta$  the contraction  $\mathcal{Y}_t \rightarrow \bar{\mathcal{Y}}_t$  is then of type I; by Section 4.3.1 there is a smoothing  $\tilde{\mathcal{Y}}_t$  of  $\bar{\mathcal{Y}}_t$  except when  $\text{Sing}(\bar{\mathcal{Y}}_t)$  is composed by a unique ordinary double point; some more technical consideration shows that the latter does not occur for general  $t$ ;

- (5) it remains to understand what happens when  $g(C) = 0$  i.e.  $C \cong \mathbb{P}^1$ ; the goal is *to construct a deformation  $\bar{\mathcal{Y}} \rightarrow \Delta$  of  $\bar{Y}$  such that the image of the induced map  $\Delta \rightarrow \text{Def}(\bar{Y})$  is not contained in  $\text{Im}(\text{Def}(Y) \rightarrow \text{Def}(\bar{Y}))$* ; if such a deformation exists then  $\bar{\mathcal{Y}}_t$  has  $\mathbb{Q}$ -factorial terminal singularities for general  $t \in \Delta$  [37, Lemma 1.6] and by results of Namikawa and Steenbrink [55] it suffices to guarantee the existence of a smoothing  $\tilde{\mathcal{Y}}_t$  of  $\bar{\mathcal{Y}}_t$ ; to show the existence of the deformation  $\bar{\mathcal{Y}}$  needs a careful analysis of the structure of  $\text{Def}(\bar{Y})$  and of the differential of the map  $\text{Def}(Y) \rightarrow \text{Def}(\bar{Y})$ :

- if  $E^3 \leq 6$  the cokernel of the above differential has dimension  $\geq 2$ ;  $\text{Def}(\bar{Y})$  is smooth when  $E^3 \leq 5$ ; if  $E^3 = 6$  then  $\text{Def}(\bar{Y})$  may not be smooth but it is set-theoretically defined by at

most one equation in a neighborhood of the origin of its tangent space; then the desired deformation  $\tilde{Y} \rightarrow \Delta$  exists for  $E^3 \leq 6$  [37, Theorem 1.7].

**Conclusion:** If  $T(Y, \bar{Y}, \tilde{Y})$  is a transition of type III then the associated contraction  $\phi$  fibres its exceptional divisor  $E$  as a conic bundle over the smooth curve  $C = \text{Sing}(\bar{Y})$ . Moreover  $C$  is a locus of cDV singularities of  $\bar{Y}$  and either  $g(C) \geq 1$  or  $g(C) = 0$  and  $\deg E \leq 6$ .

In particular  $\phi$  cannot fibre  $E$  as a conic bundle of degree 7 or 8 over  $\mathbb{P}^1$  [37, Theorem 0.4].

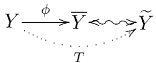
#### 4.3.4. What about a general transition?

The case of a general geometric transition  $T(Y, \bar{Y}, \tilde{Y})$  is much more complicated than the case of a primitive one, essentially for two reasons:

- the geometry of the exceptional locus  $E$  can be very intricate,
- $\bar{Y}$  can then assume very general canonical singularities so that  $\text{Def}(\bar{Y})$  can be very singular and the deformation theory of  $\bar{Y}$  very complicated.

Some partial result can be obtained from Theorem 4.18 or a generalization of it in the case of non-complete intersection singularities (see [36, Definition 4.2 and Theorem 4.3]): anyway  $\bar{Y}$  is assumed to be  $\mathbb{Q}$ -factorial and only admitting (a particular kind) of isolated singularities.

Moreover let us conclude by observing that, given a geometric transition



even the decomposition of  $\phi$  in primitive contractions can be non-invariant with respect to deformations of  $Y$ . In fact if  $\phi$  factors through a transition of type III then the Kähler cone may jump under deformation [79,80, Main Theorem; 53, Theorem C].

## 5. The Calabi–Yau web

### 5.1. Reid’s fantasy

An immediate consequence of Theorem 3.3 is that, starting from a given Calabi–Yau three-fold  $Y$ , a conifold transition produce a topologically distinct Calabi–Yau three-fold  $\tilde{Y}$ . Actually there are plenty of topologically distinct well-known examples of Calabi–Yau three-folds and this fact seems to definitely exclude the possibility of any kind of “irreducibility” for any more or less defined concept of moduli space of Calabi–Yau three-folds.

This is something new with respect to what happens in the lower dimensional cases of elliptic curves and K3 surfaces.

- *Elliptic curves.* Any one-dimensional compact complex manifold with  $K_C \equiv 0$  is biholomorphic to an algebraic smooth plane cubic curve, i.e. to a complex torus, and vice versa. In particular their complex moduli space is the moduli space of complex structures over the topological torus  $S^1 \times S^1$ . Such a moduli space is algebraic, smooth and irreducible (the well-known modular curve).
- *K3 surfaces* (see [11,4]). The following facts were known to Enriques [25]:

- $\forall g \geq 3$  there exists a  $K3$  surface of degree  $2g - 2$  in  $\mathbb{P}^8$ ; hence its sectional genus is  $g$ ;
- $\forall g \geq 3$  we can obtain a space  $\mathcal{M}_g$  of complex projective moduli of such surfaces, by imposing a polarization:  $\mathcal{M}_g$  is an irreducible, analytic variety with  $\dim_{\mathbb{C}} \mathcal{M}_g = 19$ ;
- then the complex moduli space  $\mathcal{M}^{\text{alg}}$  of algebraic  $K3$  surfaces is a *reducible* analytic variety and it admits a countable number of irreducible components;
- there exist  $K3$  surfaces belonging to more than one irreducible component of  $\mathcal{M}^{\text{alg}}$ ; anyway if we restrict to  $K3$ 's admitting  $\text{Pic} \cong \mathbb{Z}$  (they give the general element of any irreducible component) then they belong to only one irreducible component.

What could appear to Enriques as a wildly reducible moduli space was explained by Kodaira [45] as an analytic codimension 1 subvariety of a smooth, irreducible, analytic variety  $\mathcal{M}$ . More precisely:

- there exist analytic non-algebraic  $K3$  surfaces,
- the Kuranishi space of any analytic  $K3$  surface is smooth and of dimension 20.

The latter suffices to construct a smooth, irreducible, analytic universal family of  $K3$  surfaces: its base  $\mathcal{M}$  is the complex analytic moduli space of  $K3$  surfaces and  $\dim_{\mathbb{C}} \mathcal{M} = 20$ . Moreover  $\mathcal{M}^{\text{alg}}$  turns out to be a dense subset of  $\mathcal{M}$ .

In other words *the irreducibility of the moduli space of  $K3$ 's is obtained by leaving the algebraic geometric category to work in the larger category of compact, Kähler, analytic manifolds*. In fact *any  $K3$  surface is Kähler* since all of them admit a canonical Ricci flat Kähler–Einstein metric.

In [62] Reid suggested that the right approach to perceive some kind of irreducibility of a suitable moduli space of Calabi–Yau three-folds could be similar to the case of  $K3$  surfaces: one has to work in the right category. The key idea is given by the following result of Friedman.

**Theorem 5.1** ([27, Corollary 4.7]). *Let  $\phi : Y \rightarrow \bar{Y}$  be a small contraction of a Calabi–Yau three-fold  $Y$  to a normal three-fold  $\bar{Y}$  such that  $H^2(Y)$  is generated by the exceptional locus  $E$  of  $\phi$  and  $\text{Sing}(\bar{Y})$  is composed by  $N \geq 2$  nodes. Then  $\bar{Y}$  is smoothable and every smoothing  $\tilde{Y}$  has  $b_2(\tilde{Y}) = 0$ . Hence  $\bar{Y}$  can be smoothed only to non-Kähler compact complex three-folds.*

**Corollary 5.2.** *There exist “non-Kähler Calabi–Yau” three-folds which can be realized, by means of a conifold transition, starting from an algebraic Calabi–Yau three-fold  $Y$  as in Theorem 5.1.*

**Remark 5.3.** There is an evident contradiction in the words *non-Kähler Calabi–Yau* since in Definition 1.1 we assumed a projective embedding for  $Y$ . Anyway their meaning should be evident as well and probably the reader will forgive such an abuse of notation!

A “Calabi–Yau” three-fold with second Betti number equal to zero has topological type completely determined by the third Betti number. By results of Wall [76] this suffices to guarantee that it is diffeomorphic to a connected sum  $(S^3 \times S^3)^{\#r}$  of  $r$  copies of the *solid hypertorus*  $S^3 \times S^3$ . Introduce then the following assumptions.

**Assumption 5.4.** (1) Every projective Calabi–Yau three-fold  $Y$  is birational to a Calabi–Yau three-fold  $Y'$  such that  $H^2(Y')$  is generated by rational curves; moreover if  $\phi : Y' \rightarrow \bar{Y}$  is the morphism contracting all them, then  $\bar{Y}$  is always smoothable. (2) The moduli space  $\mathcal{N}_r$  of complex structures on  $(S^3 \times S^3)^{\#r}$  is *irreducible*.

Then we get the famous conjecture.

**Conjecture 5.5** (The Reid’s fantasy). *Up to some kind of inductive limit over  $r$ , the birational classes of projective Calabi–Yau three-folds can be fitted together, by means of geometric*

transitions, into one irreducible family parameterized by the moduli space  $\mathcal{N}$  of complex structures over suitable connected sum of copies of solid hypertori.

In fact if  $Y$  is a Calabi–Yau three-fold, by assumption (1) we can recover a birational Calabi–Yau three-fold  $Y'$  admitting a small contraction morphism  $\phi : Y' \rightarrow \bar{Y}$ . Since  $\phi$  is a composition of a finite number of type I contractions, Section 4.3.1 guarantees that  $\bar{Y}$  admits at most a finite number of isolated cDV singular points.

Then by Theorem 4.18,  $\bar{Y}$  can be deformed to a variety  $\bar{Y}'$  admitting at worst nodes as singularities. Recalling Theorem 4.21, the second part of assumption (1) implies that either  $|\text{Sing}(\bar{Y}')| \geq 2$  or  $\bar{Y}'$  is smooth. In the first case Theorem 4.21, or equivalently Theorem 5.1, gives a smoothing  $\tilde{Y}$  of  $\bar{Y}'$ . In the second case rename  $\bar{Y}'$  as  $\tilde{Y}$ . In both cases  $\tilde{Y}$  is a non-Kähler Calabi–Yau three-fold since  $H^2(\tilde{Y}) = 0$ . Then it is diffeomorphic to a connected sum of  $r$  copies of solid hypertori, where  $r$  depends on the topology of  $Y$ . In fact, if in particular we make the further assumption that  $\text{Sing}(\bar{Y})$  is composed only by nodes then the transition  $T(Y', \bar{Y}, \tilde{Y})$  is a conifold one and Theorem 3.3 gives

$$r = b_3(\tilde{Y})/2 = b_3(Y')/2 + c = b_3(Y')/2 + N - k$$

Assumption (1) implies that  $k = b_2(Y')$  and that  $N \geq 2$ . The previous relation can be then rewritten as follows:

$$b_3(Y') - 2b_2(Y') = 2r - N \leq 2r - 2 \tag{64}$$

Since  $Y$  and  $Y'$  are birational, their Betti numbers coincides.<sup>1</sup> Then (64) can be rewritten, in terms of the Euler–Poincaré characteristic of  $Y$ , as follows

$$r = 1 + \frac{N - \chi(Y)}{2} \geq 2 - \chi(Y)/2 \tag{65}$$

In conclusion, by means of a geometric transition, the birational equivalence class of the Calabi–Yau three-fold  $Y$  determines a complex structure over  $(S^3 \times S^3)^{\#r}$ , given by  $\tilde{Y}$  and represented by a point of  $\mathcal{N}_r$ , for  $r \gg 0$  according to (65). On the other hand, results stated in Section 3.1 ensure that the previous argument applies, with slight modifications, to any Calabi–Yau three-fold, without the assumption that  $T$  is conifold.

The last step should be a sort of *gluing* of all the  $\mathcal{N}_r$ 's preserving irreducibility postulated by assumption (2) (to use Reid's words: "let's ignore this as a minor technical problem").

**Remark 5.6.** The key point of the Reid's fantasy is clearly the assumption (2): very little is known about complex structures over solid hypertori and very few techniques are available in dealing with compact complex non-Kähler manifolds!

**Remark 5.7.** The geometric beauty of the Reid's fantasy 5.5 is given also by the evident analogies with both the lower-dimensional cases of elliptic curves and of  $K3$  surfaces. In fact as in the last case, the irreducibility of the moduli space is recovered by means of particular geometric transitions which actually are the right tools to leave the compact, Kähler category to work into

---

<sup>1</sup> This is a famous result of Batyrev [6], obtained by employing  $p$ -adic integration and Weil conjectures. It seems that this result motivated Kontsevich to introduce the theory of *motivic integration* in a memorable lecture at Orsay [47], in which he proved that two birational Calabi–Yau varieties even have isomorphic Hodge structures. Actually, as explained by Batyrev in the introduction of [6], the three-dimensional case, to which we are interested here, can be deduced by an older result of Kawamata [42], since two birational minimal models of three-folds can be connected by a sequence of flops.

the larger category of compact, complex, analytic manifolds. On the other hand, as in the case of elliptic curves, the moduli problem is reduced to parameterize complex structures over a sort of “generalized tori”.

### 5.2. The “vacuum degeneracy problem” in string theory

The geometric transition’s property of connecting topologically distinct Calabi–Yau three-folds and in particular the restored concept of a possible irreducible moduli space due to the Reid’s Conjecture 5.5 suggested most interesting applications in string theory.

In fact Calabi–Yau three-folds play a fundamental role in 10-dimensional string theories: locally four dimensions give rise to the usual Minkovsky space–time while the remaining six dimensions (the so-called *hidden dimensions* for their microscopic extension, of the same order as the Plank constant) are compactified to a geometric model which, essentially to preserve the required supersymmetry, turns out to be a Calabi–Yau three-fold.

In spite of the fact that there are very few consistent 10-dimensional super-string theories, actually near-unique via dualities, the compactification process give rise to the problem of choosing the appropriate Calabi–Yau model: on one hand there is not any prescription for making a precise choice and on the other hand there is a huge multitude of topologically distinct Calabi–Yau three-folds. Moreover the choice of two distinct Calabi–Yau models is not “a priori” equivalent from the physical point of view, since the second and the third Betti numbers (or better the Hodge numbers  $h^{1,1}$  and  $h^{2,1}$ ) of the Calabi–Yau model are strictly related with the number of hypermultiplets and the number of vector multiplets, respectively, of the compactified physical theory.

This is the so-called *vacuum degeneracy problem* in string theory.

The ideas of Clemens and then of Friedman and Reid, leading to the formulation of the Reid’s fantasy in 1987 suggested to physicists like Candelas, Green, Hübsch and others that:

- *Calabi–Yau three-folds could be, at least mathematically, connected each other by means of geometric (conifold) transitions.*

This is the so-called *Calabi–Yau web conjecture* described in many insightful papers starting from 1988 (see [17,31,32,18,19]). A more precise version of this conjecture will be given later following Gross (see 5.3).

In the previous statement *mathematically* means that the geometric (or eventually the conifold) transition connecting each other two Calabi–Yau three-folds is merely a geometrical process: what about the *physical transition* between the physical theories involved?

A first answer was given, for what concerning a conifold transition, in 1995 by Strominger (see [68,34]). His explanation of how physical theories can pass *smoothly* through the conifold singularities of the moduli space of Calabi–Yau string vacua was inspired by techniques of Seiberg and Witten [66]: the idea is that the topological change is given by the *condensation* of massive black holes to massless ones.

In the following years some other geometric transition, more general than the conifold one, have been physically understood: see for example [12,41,13].

### 5.3. The connectedness conjecture

A mathematically refined version of the Calabi–Yau web conjecture was presented by Gross [37].

On the contrary of the K3 case for which an algebraic K3 surface can be smoothly deformed to a non-algebraic one, the deformation of a projective Calabi–Yau three-fold, even singular, is still projective. Since the hardest part of Conjecture 5.5 seems to be in dealing with non-Kähler Calabi–Yau three-folds and in finding non-algebraic contractions, as observed in Remark 5.6, one could skip this part by insisting on staying within the projective category as follows.

One can think the nodes of the giant web predicted by the web conjecture as consisting in deformation classes of Calabi–Yau three-folds. Two of such nodes, say  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , are connected by an arrow  $\mathcal{M}_1 \rightarrow \mathcal{M}_2$  if the general element of  $\mathcal{M}_1$  is connected with a smooth element of  $\mathcal{M}_2$  by means of a geometric transition, which means: for the general element  $Y$  of  $\mathcal{M}_1$  there exists a birational contraction to a normal three-fold  $\phi : Y \rightarrow \bar{Y}$  and a flat local family  $\bar{\mathcal{Y}} \rightarrow \Delta$  whose central fibre is  $\bar{\mathcal{Y}}_0 \cong \bar{Y}$  and such that  $\bar{\mathcal{Y}}_t$  is a smooth element of  $\mathcal{M}_2$  for general  $t \in \Delta$ .

**Example 5.8** (See also [37]). Let

- $\mathcal{M}_Q$  be the moduli space of smooth quintic three-folds in  $\mathbb{P}^4$ ,
- $\mathcal{M}_D$  be the moduli space of double solids (i.e. double covers of  $\mathbb{P}^3$ ) branching along a smooth octic surface of  $\mathbb{P}^3$ ,
- $\mathcal{M}_T$  be the moduli space of smooth blow-up’s of quintic three-folds having a triple point.

Let  $Z$  be a general element in  $\mathcal{M}_T$  and  $\phi : Z \rightarrow \bar{Y}$  be the contraction of the exceptional divisor of  $Z$ . Then  $\bar{Y}$  is a quintic three-fold in  $\mathbb{P}^4$  with a triple point. Since  $\bar{Y}$  can be smoothed to a quintic three-fold we have

$$\mathcal{M}_T \rightarrow \mathcal{M}_Q \tag{66}$$

by means of a primitive transition of type II.

On the other hand if we project  $\bar{Y}$  from the triple point  $p_o$  we get a rational morphism

$$\psi : \bar{Y} \rightarrow \mathbb{P}^3$$

**Proposition 5.9.** *The previous rational morphism  $\psi$  can be lifted to the blow up  $Z$  giving rise to a generically finite morphism  $\hat{\psi} : Z \rightarrow \mathbb{P}^3$ . More precisely  $\hat{\psi}$  is 2:1 except over 60 points  $\{p_i\}$  for which  $\hat{\psi}^{-1}(p_i) \cong \mathbb{P}^1$ . Consider the Stein factorization  $\hat{\psi} = f \circ \varphi$ . Then we get the following commutative diagram*

$$\begin{array}{ccc}
 Z & \xrightarrow{\varphi} & \bar{X} \\
 \downarrow \phi & \searrow \hat{\psi} & \downarrow f \\
 \bar{Y} & \xrightarrow{\psi} & \mathbb{P}^3
 \end{array}
 \tag{67}$$

where  $\varphi$  is the birational contraction of all of the 60  $\mathbb{P}^1$ ’s and  $f$  gives to  $\bar{X}$  the structure of a double solid branched along a singular octic surface  $S \subset \mathbb{P}^3$ .

Since  $\bar{X}$  can immediately be smoothed by smoothing the branching locus  $S \subset \mathbb{P}^3$  it is possible to write

$$\mathcal{M}_T \rightarrow \mathcal{M}_D \tag{68}$$



- Therefore the deformation families  $\mathcal{M}_Q, \mathcal{M}_T, \mathcal{M}_D$  are nodes of the following connected graph obtained by composing (66) and (68):



**Proof of Proposition 5.8.** The rational morphism  $\psi$  is defined as follows:

$$\forall p \in \bar{Y} \setminus \{p_o\} \quad \psi(p) := l(p_o, p) \in \mathcal{L}_{p_o} := \{\text{lines } l \text{ of } \mathbb{P}^4 \text{ through } p_o\} \cong \mathbb{P}^3$$

where  $l(p_o, p)$  is the line connecting  $p$  and  $p_o$ . Since the domain  $\bar{Y} \setminus \{p_o\}$  of  $\psi$  coincides with the locus of smooth points of  $\bar{Y}$ ,  $\psi$  can be naturally lifted to a well defined morphism  $\hat{\psi} : Z \rightarrow \mathbb{P}^3$  by setting

$$\forall q \in Z \setminus E \quad \hat{\psi}(q) = (\psi \circ \phi)(q), \quad \forall q \in E \quad \hat{\psi}(q) = l_q$$

where  $E$  is the exceptional locus of the blow up  $\phi$  and  $l_q$  is the tangent line to  $\bar{Y}$  in  $p_o$  determined by the tangent direction represented by  $q \in E$ . The morphism  $\hat{\psi}$  is clearly generically 2:1 and the image in  $\mathcal{L}_{p_o} \cong \mathbb{P}^3$  of the branching locus is given by

$$S := \{\text{lines } l(p_o, p) \text{ which are tangent to } \bar{Y} \text{ in } p\} \subset \mathbb{P}^3$$

$S$  is a surface of degree 8. In fact locally the triple point  $p_o$  can be assumed to be the origin of an affine subset  $\mathbb{C}^4$  of  $\mathbb{P}^4$ . The local equation of  $\bar{Y}$  is then given by  $F_5 + F_4 + F_3 = 0$ , where  $F_d = F_d(x, y, z, w)$  is a generic homogeneous polynomial of degree  $d$ . If  $p = (x_p, y_p, z_p, w_p)$  then  $l(p_o, p)$  is parameterized by

$$x = x_p t, \quad y = y_p t, \quad z = z_p t, \quad w = w_p t$$

Therefore  $l(p_o, p) \in S$  if and only if

$$(F_5 + F_4 + F_3)|_{l(p_o, p)} = t^3(at^2 + bt + c)$$

where  $a, b, c$  are homogeneous polynomials in  $x_p, y_p, z_p, w_p$  of degree 5, 4, 3, respectively, satisfying the further tangency condition

$$b^2 - 4ac = 0 \tag{70}$$

The latter gives a degree 8 homogeneous equation in  $\mathbb{P}^3(x_p, y_p, z_p, w_p) \cong \mathcal{L}_{p_o}$ .

Observe that the 60 points  $\{p_i\}$  described in  $\mathcal{L}_{p_o}$  by  $a = b = c = 0$  are the images via  $\psi$  of the lines contained in  $\bar{Y}$ . Hence  $\hat{\psi}^{-1}(p_i) \cong \mathbb{P}^1$  while  $\hat{\psi}$  is 2:1 over  $\mathbb{P}^3 \setminus \{p_i\}$ . The Stein factorization  $\hat{\psi} = f \circ \phi$  is then the composition of the birational morphism  $\phi$  contracting all of those  $\mathbb{P}^1$ 's and of the 2:1 morphism  $f$  onto  $\mathbb{P}^3$ .

The situation is then described by the commutative diagram (67) where  $\bar{X}$  is a double covering of  $\mathbb{P}^3$  branched along the surface  $S$ . Since Eq. (70) of  $S$  gives  $\text{Sing}(S) = \{a = b = c = 0\}$ ,  $\bar{X}$  admits the 60 isolated singularities given by the images by  $\phi$  of the contracted  $\mathbb{P}^1$ 's. The smoothing of  $\bar{X}$  is then given by the double solid branched along the generic surface of degree 8 in  $\mathbb{P}^3$ .  $\square$

Let us come back to the connected graph (69). Then the question is: can that graph be enlarged to a very bigger graph connecting deformation classes of all simply connected Calabi–Yau three-folds?

**Conjecture 5.10** (Of connectedness). *The graph of simply connected Calabi–Yau three-folds is connected.*

Evidences for such a conjecture were firstly given in [32], where the moduli spaces of some Calabi–Yau three-folds, which are complete intersections in products of projective spaces, were connected each other.

Most significative evidences are given in [20] where a general procedure for connecting up Calabi–Yau three-folds which are complete intersections in some toric variety, is described. Such a procedure was developed starting from an original idea of Morrison and works by *intersecting* the combinatorial toric data (i.e. *reflexive polytopes*) of two given Calabi–Yau three-folds, to produce a further Calabi–Yau three-fold (if the so-obtained toric data give rise to a reflexive polytope too!). The latter Calabi–Yau is then connected to the previous two, by means of geometric transitions. By direct computer search, the authors checked that the procedure described allows to settle all known examples of Calabi–Yau hypersurfaces in weighted  $\mathbb{P}^4$  (7555 Calabi–Yau three-folds) into a big connected graph. This result was actually already known to Candelas and collaborators, but the new fact is that the third Calabi–Yau three-fold, obtained by *intersecting* the toric data of two given Calabi–Yau weighted hypersurfaces, is not, in general, a weighted hypersurface but rather a complete intersection in a more general toric variety. Which is: *the graph connecting up all the 7555 Calabi–Yau weighted hypersurfaces extends to englobe many complete intersections in more general toric varieties.*

Let us remark that, in general, the geometric transitions involved in the procedure described above are not conifold. Hence such a big graph produces a *mathematical* link between deformation classes of Calabi–Yau three-folds, leaving open the problem of a satisfying physical understanding of the induced connection between string vacua.

## 6. Mirror symmetry and transitions: the reverse transition

A natural question arises from the previous connectedness **Conjecture 5.10**:

- *is such a conjecture consistent with already known “connecting processes” between Calabi–Yau string vacua suggested by physical dualities like e.g. mirror symmetry?*

In a sense, a positive answer to this question represents a further evidence supporting the stated conjecture.

### 6.1. Mirror symmetry conjecture: some mathematical statements

A description of physical origin and meaning of mirror symmetry conjecture is outside the scope of this paper. In the following we will simply state some (minimal) mathematical consequences useful to understand the role of geometric transition in this context. The reader interested in a deeper understanding of the topic should consult the extensive monographs [74,22] and the recent [39].

**Conjecture 6.1** (Infinitesimal mirror symmetry). *Let  $Y$  be a Calabi–Yau variety. Then there exists a Calabi–Yau variety  $Y^\circ$  and isomorphisms of complex vector spaces*

$$\forall 0 \leq p, q \leq \dim Y \quad \mu_{p,q} : H^p(\Omega_Y^q) \xrightarrow{\cong} H^p(\Omega_{Y^\circ}^{n-q}) \tag{71}$$

*inducing a mirror reversing identification on the Hodge diamonds of  $Y$  and  $Y^\circ$ .*

**Remark 6.2.** Since  $\mathcal{K}_Y \cong \mathcal{O}_Y$  we get canonical isomorphisms

$$H^p(\Omega_Y^{n-q}) \cong H^p(\wedge^q \mathcal{T}_Y)$$

and the same for  $Y^\circ$ . Their composition with isomorphisms  $\mu_{p,q}$  in (71) give rise to the following isomorphisms

$$\forall 0 \leq p, q \leq \dim Y \quad \mu'_{p,q} : H^p(\wedge^q \mathcal{T}_Y) \xrightarrow{\cong} H^p(\Omega_{Y^\circ}^q), \quad \mu''_{p,q} : H^p(\Omega_Y^q) \xrightarrow{\cong} H^p(\wedge^q \mathcal{T}_{Y^\circ})$$

and commutative diagrams

$$\begin{array}{ccc} H^p(\wedge^{n-q} \mathcal{T}_Y) & \xrightarrow{\mu'_{p,n-q}} & H^p(\Omega_{Y^\circ}^{n-q}) \\ \downarrow \cong & \nearrow \mu_{p,q} & \downarrow \cong \\ H^p(\Omega_Y^q) & \xrightarrow{\mu''_{p,q}} & H^p(\wedge^q \mathcal{T}_{Y^\circ}) \end{array} \tag{72}$$

In particular if  $p = 1 = q$  then

$$\mu' := \mu'_{1,1} : H^1(\mathcal{T}_Y) \xrightarrow{\cong} H^1(\Omega_{Y^\circ}) \tag{73}$$

$$\mu'' := \mu''_{1,1} : H^1(\Omega_Y) \xrightarrow{\cong} H^1(\mathcal{T}_{Y^\circ}) \tag{74}$$

6.1.1. *The Calabi–Yau moduli space*

To give a Calabi–Yau variety  $Y$  means in particular to fix a triple  $(Y, J, h)$  of a compact manifold  $Y$ , a complex structure  $J$  on  $Y$  and a Hermitian metric  $h$  on  $Y$  whose real part gives a Ricci flat Riemannian metric, and whose imaginary part gives a closed  $(1, 1)$ -form  $\omega := -1/2 \operatorname{Im} h$  (i.e. a Kähler form) which is positive.

Think the complex moduli space  $\mathcal{M}_Y^{\mathbb{C}}$  of  $(Y, J, h)$  as the space parameterizing all the deformations of the complex structure  $J$  over  $Y$  up to biholomorphisms. The Bogomolov–Tian–Todorov theorem asserts that locally  $\mathcal{M}_Y^{\mathbb{C}}$  is smooth (see [14,71,73] and also [58] for a more recent and algebraic proof). Then:

- $H^1(\mathcal{T}_Y)$  can be canonically identified with the tangent space to  $\mathcal{M}_Y^{\mathbb{C}}$  at the fixed complex structure  $J$ .

On the other hand the Yau theorem solving the Calabi conjecture (see [16,82]) ensures that, for any positive Kähler form  $\omega$  such that  $[\omega] \in H^2(Y, \mathbb{R}) \cap H^1(\Omega_Y)$ , there exists a unique Ricci flat metric whose associated  $(1,1)$ -form is cohomologous to  $\omega$ . Then Definition 4.3 and Corollary 4.6 imply that all the possible deformations of the Ricci flat, Kähler metric  $h$  on  $Y$  are parameterized by the Kähler cone  $\mathcal{K}(Y)$ . For this reason  $H^1(\Omega_Y)$  can be thought as a complexification of the tangent space to the Kähler moduli space of  $Y$ .

Moreover one can give a more natural meaning to  $H^1(\Omega_Y)$  by constructing a complexified Kähler moduli space as follows.

First of all observe that the mathematical datum of a given Calabi–Yau three-fold  $Y$ , which is actually a triple  $(Y, J, h)$  with  $\dim Y = 3$ , do not completely characterize the physical string theory compactified to  $Y$ . To do this an extra-datum, called the *B-field*, is needed. Physically it is a characteristic parameter of the string action. Mathematically it is represented by the choice of a lateral class  $\beta$  in the quotient  $H^2(Y, \mathbb{R})/H^2(Y, \mathbb{Z})$ . One can then look at the complex class

$$\chi := \beta + i\omega = \beta - i/2 \operatorname{Im} h \in H^2(Y, \mathbb{C})/H^2(Y, \mathbb{Z})$$

where  $H^2(Y, \mathbb{Z})$  acts naturally by inclusion, hence it acts only on the real part of a class in  $H^2(Y, \mathbb{C})$ , as desired. The class  $\chi$  is called the *complexified Kähler class* of the Calabi–Yau three-fold  $Y$ . It can be actually thought as a *polarization* over the complex variety  $(Y, J)$  whose possible deformations are then parameterized by the *complexified Kähler space*

$$\mathcal{K}_{\mathbb{C}}(Y) := \left\{ \chi \in H^2(Y, \mathbb{C}) \mid \text{Im } \chi \in \mathcal{K}(Y) \right\} / H^2(Y, \mathbb{Z}).$$

The *complexified Kähler moduli space*  $\mathcal{M}_Y^K$  is then given by  $\mathcal{K}_{\mathbb{C}}(Y)$  up to the action of the automorphisms group  $\text{Aut}(Y)$  and

- $H^1(Y, \Omega_Y)$  is the *tangent space* to  $\mathcal{M}_Y^K$  at the fixed complexified Kähler class  $\chi$ ,
- the *Calabi–Yau moduli space* of a Calabi–Yau variety  $Y$  is then the total space of a fibration

$$\mathcal{M}_Y \rightarrow \mathcal{M}_Y^{\mathbb{C}} \tag{75}$$

whose fibre over the isomorphism class in  $\mathcal{M}_Y^{\mathbb{C}}$  represented by  $(Y, J)$  is given by the complexified Kähler moduli space  $\mathcal{M}_Y^K$ .

In particular, if  $\dim Y = 3$ , Wilson proved that, outside of a countable union of closed subsets of  $\mathcal{M}_Y^{\mathbb{C}}$ , the Kähler cone do not varies with the complex structure  $J$  (see [79,80]). That is enough to conclude that:

- if  $\dim Y = 3$  the fibration (75) is *generically locally trivial*, which means that if  $J$  is the complex structure of a sufficiently general Calabi–Yau three-fold  $Y$  then there exists a Zariski open subset  $U \subset \mathcal{M}_Y^{\mathbb{C}}$  containing the class represented by  $(Y, J)$  and such that  $\mathcal{M}_Y|_U \cong U \times \mathcal{M}_Y^K$ .

Conjecture 6.1 can then be understood as the *differential version* of the following one.

**Conjecture 6.3** (Local mirror symmetry for Calabi–Yau three-fold). *Let  $(Y, \chi)$  be the polarized couple given by a general Calabi–Yau three-fold  $Y = (Y, J, h)$  and a complexified Kähler class  $\chi \in \mathcal{K}_{\mathbb{C}}(Y)$  such that  $\text{Im } \chi = -1/2 \text{Im } h$ . Then there exist:*

- (1) a mirror polarized couple  $(Y^\circ, \chi^\circ)$ , where  $Y^\circ = (Y^\circ, J^\circ, h^\circ)$  is a sufficiently general Calabi–Yau three-fold and  $\chi^\circ \in \mathcal{K}_{\mathbb{C}}(Y^\circ)$  is such that  $\text{Im } \chi^\circ = -1/2 \text{Im } h^\circ$ ,
- (2) two open subsets  $\mathcal{U} \subset \mathcal{M}_Y, \mathcal{U}^\circ \subset \mathcal{M}_{Y^\circ}$  containing the isomorphisms classes represented by  $(Y, \chi)$  and  $(Y^\circ, \chi^\circ)$ , respectively; notice that they inherits the local product structure of  $\mathcal{M}_Y$  and  $\mathcal{M}_{Y^\circ}$  i.e.

$$\mathcal{U} \cong U_{\mathbb{C}} \times U_K, \quad \mathcal{U}^\circ \cong U_{\mathbb{C}}^\circ \times U_K^\circ$$

- (3) a biholomorphism  $m : \mathcal{U} \rightarrow \mathcal{U}^\circ$ , called *local mirror map*, reversing the product structures, which is

$$m(U_{\mathbb{C}}) = U_K^\circ, \quad m(U_K) = U_{\mathbb{C}}^\circ$$

whose differential gives maps  $\mu'$  and  $\mu''$  in (73), i.e.

$$d_{(J,\chi)}(m) = \mu' \times \mu''$$

**Remark 6.4** (Mirror partners of rigid Calabi–Yau varieties). Let  $Y$  be a *rigid* Calabi–Yau variety i.e.  $Y$  do not admits complex deformations and

$$h^1(\mathcal{T}_Y) = h^{2,1}(Y) = 0 \tag{76}$$

Assume  $Y^\circ$  to be a mirror partner of  $Y$ . Then **Conjecture 6.1** gives

$$h^{1,1}(Y^\circ) = h^{2,1}(Y) = 0 \tag{77}$$

which implies that  $Y^\circ$  cannot be a Kähler variety: in particular  $Y^\circ$  is not a Calabi–Yau variety.

Since rigid Calabi–Yau three-folds exist (the first examples were constructed in 1986 by Schoen [65]) this fact introduces a counterexample to both the stated mirror symmetry **Conjectures 6.1 and 6.3**.

From the mathematical point of view, such a contradiction could be resolved by assuming mirror symmetry to involve some *non-Kähler Calabi–Yau* variety too (recall **Remark 5.6**): but which of them?

Anyway, from the physical point of view, it is completely unclear which kind of string theory can be compactified to a non-Kähler Calabi–Yau three-fold: so what is the mirror dual of a string theory compactified to a rigid Calabi–Yau vacuum?

### 6.2. The reverse transition

Consider a transition  $T(Y, \bar{Y}, \tilde{Y})$  and let  $Y^\circ$  and  $\tilde{Y}^\circ$  be mirror partners of  $Y$  and  $\tilde{Y}$ , respectively:

$$\begin{array}{ccccc}
 & & T & & \\
 & & \curvearrowright & & \\
 Y & \xrightarrow{\phi} & \bar{Y} & \xleftarrow{\quad} & \tilde{Y} \\
 \uparrow & & & & \uparrow \\
 & M.S. & & & M.S. \\
 Y^\circ & & & & \tilde{Y}^\circ \\
 \downarrow & & & & \downarrow
 \end{array} \tag{78}$$

Recall that mirror symmetry exchange complex moduli with Kähler moduli. On the other hand, if  $T$  is a conifold transition, point (3) of **Theorem 3.3** and **Remark 3.5** allow to conclude that the topologies of  $Y^\circ$  and  $\tilde{Y}^\circ$  are compatible with a (reverse) conifold transition  $T^\circ(\tilde{Y}^\circ, \bar{Y}^\circ, Y^\circ)$  which would complete diagram (78) as follows

$$\begin{array}{ccccc}
 & & T & & \\
 & & \curvearrowright & & \\
 Y & \xrightarrow{\phi} & \bar{Y} & \xleftarrow{\quad} & \tilde{Y} \\
 \uparrow & & & & \uparrow \\
 & M.S. & & & M.S. \\
 Y^\circ & \xleftarrow{\phi^\circ} & \bar{Y}^\circ & \xleftarrow{\quad} & \tilde{Y}^\circ \\
 \downarrow & & & & \downarrow \\
 & & T^\circ & & \\
 & & \curvearrowleft & & 
 \end{array} \tag{79}$$

Notice that the *reverse conifold transition*  $T^\circ$  would have the same parameters  $N, k, c$  as  $T$  whose role is now reversed. Precisely

- $\text{Sing}(\tilde{Y}^\circ)$  would be composed by  $N$  ordinary double points, just like  $\text{Sing}(\tilde{Y})$ ,
- the exceptional locus of the birational contraction  $\phi^\circ$  would be composed by  $N$  rational curves whose homology classes span a  $c$ -dimensional subspace of  $H_2(\tilde{Y}^\circ)$ ,
- the vanishing locus of the smoothing  $Y^\circ$  would be given by  $N$  three-spheres whose homology classes span a  $k$ -dimensional subspace of  $H_3(Y^\circ)$ .

A similar picture naturally suggested that a diagram like (79) could be established for every geometric transition  $T$ , leading to the following conjecture, probably due to Morrison.

**Conjecture 6.5** (Of reverse transition, see [51,34,20,48]). *Let  $T(Y, \bar{Y}, \tilde{Y})$  be a geometric transition and let  $Y^\circ$  and  $\tilde{Y}^\circ$  be mirror partners of  $Y$  and  $\tilde{Y}$ , respectively. Then mirror partners are linked by a reverse geometric transition  $T^\circ(\tilde{Y}^\circ, \bar{Y}^\circ, Y^\circ)$  like in diagram (79).*

In [51] Morrison supported such a conjecture with an example employing the Greene–Plesser construction [33] to produce mirror partners of the geometric transition linking a desingularization of an octic weighted hypersurface of  $\mathbb{P}(1, 1, 2, 2, 2, 2)$  with the generic complete intersection of bi-degree  $(2, 4)$  in  $\mathbb{P}^5$ .

Further evidences were given in [8] where the reverse transition of a conifold transition, linking a complete intersection in a Grassmannian with a complete intersection in a Fano toric variety, is produced: in particular the reverse transition is still conifold. This fact suggests to specialize Conjecture 6.5 as follows.

**Conjecture 6.6.** *Let  $T(Y, \bar{Y}, \tilde{Y})$  be a conifold transition. Then there exist mirror partners  $Y^\circ$  and  $\tilde{Y}^\circ$  of  $Y$  and  $\tilde{Y}$  and a reverse transition  $T^\circ(\tilde{Y}^\circ, \bar{Y}^\circ, Y^\circ)$  which is still conifold.*

Such a conjecture seems to be natural when we look at the role played by parameters  $N, k, c$ . Anyway in [41] examples of geometric *non-conifold* transitions  $T(Y, \bar{Y}, \tilde{Y})$ , which can be deformed to conifold transitions, are produced. More precisely the birational contraction  $\phi : Y \rightarrow \bar{Y}$  is a composition of type III birational contractions whose exceptional divisors  $E_1, \dots, E_k$  are contracted down to a unique smooth irreducible curve  $C$  of compound Du Val singularities of type  $cA_k$ . Examples given in [41] are three-dimensional hypersurfaces or complete intersections in weighted projective spaces where birational contractions  $\phi$ 's are induced by morphisms globally defined between the weighted projective spaces. For each example a *non-toric* deformation direction for  $Y$  is exhibited, producing a deformation  $\phi' : Y' \rightarrow \bar{Y}'$  of  $\phi$  which is now a *small* birational contraction (a composition of type I contractions). Moreover  $\text{Sing}(\bar{Y}')$  turns out to be composed only by nodes. Then  $T$  deforms to a conifold transition  $T'(Y', \bar{Y}', \tilde{Y}')$  as follows:



In particular  $E_1, \dots, E_k$  are deformed to

$$\binom{k+1}{2}$$

collections of  $2g - 2$  homologous rational curves in  $Y'$ , where  $g$  is the genus of  $C$ , and  $C$  is deformed to

$$N = \binom{k+1}{2} (2g - 2)$$

nodes in  $\bar{Y}'$ . Since  $E_1, \dots, E_k$  span a  $k$ -dimensional subspace of  $H_4(Y)$ , the  $N$  rational curves in  $Y'$  span a  $k$ -dimensional subspace of  $H_2(Y')$ . Setting  $c = N - k$  one can then recover parameters  $N, k, c$  for the given non-conifold transition  $T$ . If  $T'$  admits a reverse conifold transition  $T'^{\circ}$  (of parameters  $N, c, k$ ), as [Conjecture 6.6](#) predicts, then the latter admits also  $T$  as reverse transition. Therefore:

- it can happen that a conifold transition of parameters  $N, k, c$  admits a non-conifold reverse transition whose birational morphism contracts  $c$  exceptional divisors down to a smooth irreducible curve of genus

$$g = 1 + \frac{N}{2 \binom{c+1}{2}}$$

This fact do not contradicts [Conjecture 6.6](#) if the following one is true.

**Conjecture 6.7.** A geometric transition  $T(Y, \bar{Y}, \tilde{Y})$  satisfying some good condition (e.g. such that  $\phi$  contracts  $k$  exceptional divisors down to a smooth curve of genus  $g > 1$  whose points are  $cA_k$  singularities) can be deformed to a conifold transition  $T'(Y', \bar{Y}', \tilde{Y})$  like in diagram (80).

### 6.3. Toric degenerations: conifold transitions to construct mirror manifolds

Methods in [8] were generalized in [9] to complete intersections in partial flag manifolds giving a conjectural approach to produce examples verifying [Conjecture 6.6](#). On the other hand their method describes a conjectural procedure to generalize the mirror construction for Calabi–Yau complete intersections in toric Fano varieties, given in [5,10,15], to the case of Calabi–Yau complete intersections in non-toric Fano varieties. A further generalization of this construction is given in [7]. Main ideas are the following.

**Definition 6.8** ([7, Definition 3.1]). Let  $X \subset \mathbb{P}^m$  be a smooth Fano variety of dimension  $n$ . A normal Gorenstein toric Fano vareity  $P \subset \mathbb{P}^m$  is called a *small toric degeneration of X*, if there exists a Zariski open neighborhood  $U$  of  $0 \in \mathbb{C}$  and an irreducible subvariety  $\mathcal{X} \subset \mathbb{P}^m \times U$  such that the morphism  $\pi : \mathcal{X} \rightarrow U$  is flat and the following conditions hold:

- (1) the fibre  $X_t := \pi^{-1}(t) \subset \mathbb{P}^m$  is smooth for all  $t \in U \setminus \{0\}$ ;
- (2) the special fibre  $X_0 := \pi^{-1}(0) \subset \mathbb{P}^m$  has at worst Gorenstein terminal singularities and  $X_0 \cong P$ ;
- (3) the canonical homomorphism  $\text{Pic}(\mathcal{X}/U) \rightarrow \text{Pic}(X_t)$  is an isomorphism for all  $t \in U$ .

#### Example 6.9.

- (1) In [8] it is shown that the Grassmannian  $X := \mathbb{G}(r, s)$ , embedded in  $\mathbb{P}^{\binom{s}{r}-1}$  by the usual Plücker embedding, admits a small toric degeneration  $P := P(r, s) \subset \mathbb{P}^{\binom{s}{r}-1}$ .

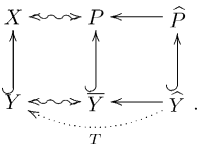
- (2) In [9] it is proved that the partial flag manifold  $X := F(n_1, \dots, n_k, n)$  with its Plücker embedding in  $\mathbb{P}^m$  admits a small toric degeneration  $P \subset \mathbb{P}^m$ .
- (3) In [7] the toric hypersurface  $P$ , given by the following homogeneous equation of degree  $d$  in  $\mathbb{P}^n$

$$z_1 \cdots z_d = z_{d+1} \cdots z_{2d}$$

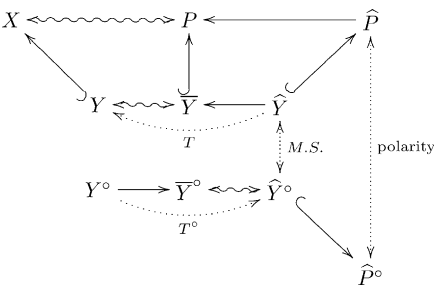
where  $n \geq 2d - 2$ , is proved to be a small toric degeneration of the generic smooth Fano hypersurface  $X$  of degree  $d$  in  $\mathbb{P}^n$ .

**Remark 6.10.** For all the previous examples,  $\text{Sing}P$  has codimension at least 3. Moreover the codimension 3 part of  $\text{Sing}P$  consists of ordinary double points.

Let now  $H$  be a generic complete intersection in  $\mathbb{P}^m$  cutting on a smooth Fano variety  $X \subset \mathbb{P}^m$  a smooth Calabi–Yau variety  $Y$ . If  $X$  admits a small toric degeneration  $P \subset \mathbb{P}^m$  and  $\hat{Y} := H \cap P$  then  $\text{Sing} \hat{Y}$  has codimension at least 3. In particular if  $\dim Y = 3 = \dim \hat{Y}$  then  $\text{Sing} \hat{Y}$  consists only of isolated nodes. Let  $\hat{P}$  be a simultaneous desingularization of  $P$  given by a suitable subdivision of the fan associated with  $P$ . Then the birational morphism  $\hat{P} \rightarrow P$  induces a desingularization  $\hat{Y} \rightarrow Y$ . We have then a geometric transition  $T(\hat{Y}, \hat{Y}, Y)$  which is conifold when  $\dim Y = 3$



The mirror partner of  $\hat{Y}$  given by the construction of [10,15], is a complete intersection  $\hat{Y}^\circ$  in the dual Fano toric variety  $\hat{P}^\circ$  obtained by polarity on associate polytopes. The main point is that the embedding  $\text{Pic} P \hookrightarrow \text{Pic} \hat{P}$  suggests, via monomial-divisor correspondence [3], a canonical way to specialize  $\hat{Y}^\circ$  to a singular  $\hat{Y}^\circ$ . Let  $Y^\circ \rightarrow \hat{Y}^\circ$  be a minimal desingularization. The situation is then the following



and  $Y^\circ$  is conjectured to be a mirror partner of  $Y$  and  $T^\circ$  be a reverse transition of  $T$ . In particular for all the given three-dimensional examples verifying this conjecture (see [8])  $T^\circ$  turns out to be a conifold transition like  $T$ .

6.4. Mirror partners of rigid Calabi–Yau three-folds via geometric transitions

Let  $Y$  be a rigid Calabi–Yau three-fold as in Remark 6.4. At least from the mathematical point of view, the reverse transition Conjecture 6.5 gives an answer to which non-Kähler Calabi–Yau



three-fold  $Y^\circ$  should be a mirror partner of  $Y$ . In fact

- if there exists a geometric transition  $T(Y, \bar{Y}, \tilde{Y})$  then  $h^{2,1}(\tilde{Y}) > h^{2,1}(Y) = 0$ , since  $\tilde{Y}$  cannot be rigid,
- let  $\tilde{Y}^\circ$  be a mirror partner of  $\tilde{Y}$  and  $T^\circ(\tilde{Y}^\circ, \bar{Y}^\circ, Y^\circ)$  be a reverse transition of  $T$ ,
- then  $Y^\circ$  should be a mirror partner of  $Y$  like in diagram (79).

If  $T$  and  $T^\circ$  are both conifold then, from the physical point of view, the previous procedure suggests that the mirror dual of a string theory compactified to a rigid Calabi–Yau three-fold can be obtained by a suitable composition of black hole condensations and mirror symmetry (over non-rigid Calabi–Yau three-folds).

### 7. Further physical dualities and transitions

The local conifold transition

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\quad} \bar{U} \begin{matrix} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{matrix} T^*S^3$$

$\xrightarrow[\text{---}]{T}$

(81)

studied in Section 2, has been recently considered as the geometric set up of a new conjectured open/closed string duality.

More precisely, at the beginning, in 1974, t’Hooft conjectured that large  $N$  gauge theories are dual to closed string theories [70]. Later, in 1992, Witten showed that a particular kind of gauge theory, namely a  $SU(N)$  (or  $U(N)$ ) Chern–Simons gauge theory on the three-sphere  $S^3$ , is equivalent to an open string theory on  $T^*S^3$  with  $D$ -branes wrapped on  $S^3$  [81]. In 1998, Gopakumar and Vafa conjectured that, for large  $N$ , a  $SU(N)(U(N))$  Chern–Simons gauge theory is dual to a closed string theory “compactified” to the local Calabi–Yau three-fold  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  [29,56]. Composing all these dualities gives an open/closed string duality modelled on the local conifold transition (81). For all the details, the interested reader is referred to original papers, and to [30] for a survey on these topics and more references.

The concept of reverse transition, introduced in the previous section, applied to such an open/closed string duality, suggests a further duality on the mirror theories. This was proposed in [2].

Examples of similar dualities, geometrically realized by less elementary conifold transitions than (81), are given in [23]. A reverse transition of one of them is described in the recent paper [26].

### Acknowledgements

I would like to especially thank A. Collino and A. Grassi for many suggestions and stimulating discussions. In particular A. Grassi introduced me to this kind of problems. Thanks are also due to M. Billó and I. Pesando for many interesting discussions about string theory.

### References

[1] K. Altman, The versal deformation of an isolated toric Gorensteins singularity, *Invent. Math.* 128 (1997) 443–479. math.AG/9403004.  
 [2] M. Aganagic, C. Vafa,  $G_2$  mainfolds, mirror symmetry and geometric engineering. hep-th/0110171.

- [3] P. Aspinwall, B. Greene, D. Morrison, The monomial-divisor mirror map, *Int. Math. Res. Not.* 12 (1993) 319–337. math.AG/9309007.
- [4] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, vol. 4, E.M.G., Springer-Verlag, 1984.
- [5] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties, *J. Alg. Geom.* 3 (1994) 493–535. math.AG/9310003.
- [6] V. Batyrev, Birational Calabi–Yau  $n$ -folds have equal Betti numbers, in: *New trends in Algebraic Geometry*, Warwick 1996, London Math. Soc. Lecture Note Ser. 264, Cambridge University Press, Cambridge, 1999, pp. 1–11. math.AG/9710020.
- [7] V. Batyrev, Toric degenerations of Fano varieties and constructing mirror manifolds, in: A. Collino, A. Conte, M. Marchisio (Eds.), *Proceedings of the Fano Conference*, Torino, 2002, Dip. di Mat. dell’Univ. di Torino, Torino, 2004, pp. 109–122. math.AG/9712034.
- [8] V. Batyrev, I. Ciocan-Fontanine, B. Kim, D. van Straten, Conifolds transitions and mirror symmetry for Calabi–Yau complete intersection in Grassmannians, *Nucl. Phys. B* 154 (1998) 640–666. math.AG/9710022.
- [9] V. Batyrev, I. Ciocan-Fontanine, B. Kim, D. van Straten, Mirror symmetry and toric degenerations of partial flag manifolds, *Acta Math.* 184 (2000) 1–39. math.AG/9803108.
- [10] V. Batyrev, D. van Straten, Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties, *Commun. Math. Phys.* 168 (1995) 493–533. math.AG/9307010.
- [11] A. Beauville, *Surfaces algébrique complexes.*, Astérisqué 54 (1978).
- [12] P. Berglund, S. Katz, A. Klemm, Mirror Symmetry and the moduli space for generic hypersurfaces in toric varieties, *Nucl. Phys. B* 456 (1995) 153–204. hep-th/9506091.
- [13] P. Berglund, S. Katz, A. Klemm, P. Mayr, New Higgs transition between  $N = 2$  string models., *Nucl. Phys. B* 483 (1997) 209–228. hep-th/9605154.
- [14] F. Bogomolov, Hamiltonian Kähler manifolds, *Dokl. Akad. Nauk. SSSR* 243 (5) (1978) 1101–1104.
- [15] L.A. Borisov, Towards the mirror symmetry for Calabi–Yau complete intersections in toric varieties. math.AG/9310001.
- [16] E. Calabi, The space of Kähler metrics, in: *Proceedings of the International Congress of Mathematicians*, vol. 2, Amsterdam, 1954, North-Holland, Amsterdam, 1956, pp. 206–207.
- [17] P. Candelas, A.M. Dale, C.A. Lütken, R. Schimmrigk, Complete intersection Calabi–Yau manifolds, *Nucl. Phys. B* 298 (1988) 493–525.
- [18] P. Candelas, P.S. Green, T. Hübsch, Finite distance between distinct Calabi–Yau vacua, *Phys. Rev. Lett.* 62 (1989) 1956–1959.
- [19] P. Candelas, P.S. Green, T. Hübsch, Rolling among Calabi–Yau vacua, *Nucl. Phys. B* 330 (1990) 49–102.
- [20] T.M. Chiang, B. Greene, M. Gross, Y. Kanter, Black Hole condensation and the web of Calabi–Yau manifolds, *Nucl. Phys. Proc. Suppl.* 46 (1996) 82–95. hep-th/9511204.
- [21] C.H. Clemens, Double Solids, *Adv. Math.* 47 (1983) 107–230.
- [22] A.D. Cox, S. Katz, *Mirror Symmetry and Algebraic Geometry*, vol. 68: Math. Surveys and Monographs, American Mathematical Society, Providence, RI, 1999.
- [23] D.E. Diaconescu, B. Florea, A. Grassi, Geometric transitions and open string instantons, *Adv. Theor. Math. Phys.* 6 (2003) 619–642. hep-th/0205234.
- [24] A. Dimca, Betti numbers of hypersurfaces and defects of linear systems, *Duke Math. J.* 60 (1990) 285–298.
- [25] F. Enriques, *Le superficie algebriche*, Zanichelli, Bologna, 1946.
- [26] B. Forbes, Computations on  $B$ -model geometric transitions. hep-th/0408167.
- [27] R. Friedman, Simultaneous resolution of threefold double points, *Math. Ann.* 247 (1986) 671–689.
- [28] R. Friedman, On threefolds with trivial canonical bundles, in: *Complex Geometry and Lie Theory*, Sundance, UT, 1989, Proc. Sympos. Pure Math. 53, American Mathematical Society, 1991, pp. 103–134.
- [29] R. Gopakumar, C. Vafa, On the gauge theory/geometry correspondence, *Adv. Theor. Math. Phys.* 3 (1999) 1415–1443. hep-th/9811131.
- [30] A. Grassi, M. Rossi, Large  $N$ -dualities and transitions in geometry, in: *Geometry and Physics of Branes*, Como 2001 Series of H.E.P., Cosmology and Gravitation, IoP Bristol, 2003, pp. 210–278. math.AG/0209044.
- [31] P.S. Green, T. Hübsch, Possible phase transitions among Calabi–Yau compactifications, *Phys. Rev. Lett.* 61 (1988) 1163–1166.
- [32] P.S. Green, T. Hübsch, Connetting moduli spaces of Calabi–Yau threefolds, *Commun. Math. Phys.* 119 (1988) 431–441.
- [33] B. Greene, M.R. Plesser, Duality in Calabi–Yau moduli space, *Nucl. Phys. B* 338 (1990) 15–37.
- [34] B. Greene, D.R. Morrison, A. Strominger, Black hole condensation and the unification of string vacua, *Nucl. Phys. B* 451 (1995) 109–120. hep-th/9504145.

- [35] P.A. Griffiths, On the periods of certain rational integrals, I, II, *Ann. Math.* 90 (1969) 460–541.
- [36] M. Gross, Deforming Calabi–Yau threefolds, *Math. Ann.* 308 (1997) 187–220. [math.AG/9506022](#).
- [37] M. Gross, Primitive Calabi–Yau threefolds, *J. Diff. Geom.* 45 (1997) 288–318. [math.AG/9512002](#).
- [38] F. Hirzebruch, Some examples of threefolds with trivial canonical bundle, in: *Collected Papers*, vol. II, Springer-Verlag, 1987, pp. 757–770.
- [39] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow, *Mirror Symmetry*, vol. 1: *Clay Math. Monographs*, American Mathematical Society, Providence, RI, 2003.
- [40] D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Science Publications, Oxford, 2000.
- [41] S. Katz, D.R. Morrison, M. Plesser, Enhanced gauge symmetry in type II string theory, *Nucl. Phys. B* 477 (1996) 105–140. [hep-th/9601108](#).
- [42] Y. Kawamata, Crepant blowing ups of three dimensional canonical singularities and applications to degenerations of surfaces, *Ann. Math.* 127 (1988) 93–163.
- [43] S. Kleiman, Toward a numerical theory of ampleness, *Ann. Math.* 84 (1966) 293–344.
- [44] H. Kleppe, D. Laksov, The algebraic structure and deformation of Pfaffian scheme, *J. Algebra* 64 (1980) 167–189.
- [45] K. Kodaira, On the structure of compact complex analytic surfaces, *Am. J. Math.* 86 (1964) 751–798.
- [46] J. Kollár, S. Mori, Classification of three-dimensional flips, *J. Am. Math. Soc.* 5 (1992) 533–703.
- [47] M. Kontsevich, *Motivic integration Lecture at Orsay*, December 7, 1995.
- [48] A. Lynker, R. Schimmrigk, Conifold transitions and mirror symmetries, *Nucl. Phys. B* 484 (1997) 562–582. [hep-th/9511058](#).
- [49] J. Milnor, *Singular points of complex hypersurfaces*, in: *Annals of Mathematical Studies* 61, Princeton University Press, Princeton, 1968.
- [50] S. Mori, Threefolds whose canonical bundles are not numerically effective, *Ann. Math.* 116 (1982) 133–176.
- [51] D.R. Morrison, Through the looking glass, in: *Mirror Symmetry III*, American Mathematical Society and International Press, 1999, pp. 263–277. [math.AG/9705028](#).
- [52] D.R. Morrison, N. Seiberg, External transitions and five-dimensional supersymmetric field theories, *Nucl. Phys. B* 483 (1997) 229–247. [hep-th/9609070](#).
- [53] Y. Namikawa, On deformation of Calabi–Yau with terminal singularities, *Topology* 33 (1994) 429–446.
- [54] Y. Namikawa, Stratified local moduli of Calabi–Yau 3-folds, *Topology* 41 (2002) 1219–1237.
- [55] Y. Namikawa, J. Steenbrink, Global smoothing of Calabi–Yau 3-fold, *Invent. Math.* 122 (1995) 403–419.
- [56] H. Ooguri, C. Vafa, Knot invariants and topological strings, *Nucl. Phys. B* 577 (2000) 419–438. [hep-th/9912123](#).
- [57] Z. Ran, Deformations of maps, in: E. Ballico, C. Ciliberto (Eds.), *Algebraic Curves and Projective Geometry*, *Lecture Notes in Mathematics* 1389, Springer-Verlag, 1989.
- [58] Z. Ran, Deformations of manifolds with torsion or negative canonical bundle, *J. Alg. Geom.* 1 (1992) 279–291.
- [59] M. Reid, Canonical 3-folds, in: *Journées de géométrie algébrique d’Angers*, Sijthoff & Norddhoff, 1980, pp. 671–689.
- [60] M. Reid, Minimal model of canonical 3-folds., in: *Algebraic Varieties and Analytic Varieties*, *Adv. Stud. Pure Math.* 1, North-Holland, 1983, pp. 131–180.
- [61] M. Reid, Young person’s guide to canonical singularities, in: *Algebraic Geometry*, Bowdoin 1985, vol. 1, *Proc. Sym. Pure Math.* 46, AMS, 1987, pp. 354–414.
- [62] M. Reid, The moduli space of 3-folds with  $K = 0$  may nevertheless be irreducible, *Math. Ann.* 287 (1987) 329–334.
- [63] M. Reid, Non-normal del Pezzo surfaces, *Math. Proc. RIMS* 30 (1994) 695–727.
- [64] M. Schlessinger, Rigidity of quotient singularities, *Invent. Math.* 14 (1971) 17–26.
- [65] C. Schoen, Complex multiplication cycles on elliptic modular threefolds, *Duke Math. J.* 53 (1986) 771–794.
- [66] N. Sieberg, E. Witten, Electromagnetic duality, monopole condensation and confinement in  $N = 2$  supersymmetric Yang–Mills theory, *Nucl. Phys. B* 426 (1994) 19–52. [hep-th/9407087](#).
- [67] J.H.M. Steenbrink, Vanishing theorems on singular spaces, *Asterisque* 130 (1985) 330–341.
- [68] A. Strominger, Massless black holes and conifolds in string theory, *Nucl. Phys. B* 451 (1995) 97–109. [hep-th/9504145](#).
- [69] C. Tamás, On the classification of crepant analytically extremal contractions of smooth three-folds, *Compos. Math.* 140 (2004) 1561–1578. [math.AG/0204015](#).
- [70] G. t’Hooft, A planar diagram theory for strong interactions, *Nucl. Phys. B* 72 (1974) 461–473.
- [71] G. Tian, Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Weil–Petersson metric, in: S.-T. Yau (Ed.), *Mathematical Aspects of String Theory*, World Scientific, Singapore, 1987, pp. 629–646.
- [72] G. Tian, Smoothing threefold with trivial canonical bundle and ordinary double points, in: *Essays on Mirror Manifolds Internat*, Hong Kong Press, 1992, pp. 458–479.
- [73] A. Todorov, The Weil–Petersson geometry of the moduli space of  $SU(n \geq 3)$  (Calabi–Yau) manifolds, *Commun. Math. Phys.* 126 (1989) 325–346.
- [74] C. Voisin, *Symétrie Miroir Panoramas at Synthèses 2*, Soc. Math. France (1996).

- [75] B. Van Geemen, J. Werner, New examples of threefolds with  $c_1 = 0$ , *Math. Z.* 203 (1990) 211–225.
- [76] C.T.C. Wall, Classification problems in topology V: on certain 6-manifolds, *Invent. Math.* 1 (1966) 355–374.
- [77] J. Werner, Kleine Auflösungen spezieller dreidimensionaler Varietäten, *Bonn. Math. Schr.* (1987) 186.
- [78] P.M.H. Wilson, Calabi–Yau manifolds with large Picard number, *Invent. Math.* 98 (1989) 139–155.
- [79] P.M.H. Wilson, The Kähler cone on Calabi–Yau threefolds, *Invent. Math.* 107 (1992) 561–583.
- [80] P.M.H. Wilson, Erratum to “The Kähler cone on Calabi–Yau threefolds”, *Invent. Math.* 114 (1993) 231–233.
- [81] E. Witten, Chern–Simons gauge theory as a string theory, in: *The Floer Memorial Volume*, Birkhäuser, 1995, pp. 637–678. [hep-th/9207094](#).
- [82] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equations, I, *Commun. Pure Appl. Math.* 31 (1978) 339–411.